PART A

Unit - I

Fourier Series

1.1 Introduction

In Differential Calculus we are familiar with the expansion of a differentiable function $f(x)$ in the form of a power series. Taylor's series of $f(x)$ about $x = a$ is an infinite series in ascending powers of $(x - a)$ and Maclaurin's series is an infinite series in ascending powers of x. In many engineering problems it becomes necessary to expand a given function $f(x)$ in a series containing cosine and sine terms which belongs to a class of functions called periodic functions.

In this unit we discuss various aspects of such series referred to as ***Fourier series.***

As a preamble we briefly present two concepts connected with infinite sereies of positive terms.

1.2 Convergence and Divergence of infinite series of positive terms

If u_n is a function of n defined for all integral values of n, an expression of the form $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ containing infinite number of terms is called an ***infinite series*** usually denoted by $\sum_{n=1}^{\infty} u_n$ or simply $\sum u_n$.

u_n is called the n^{th} term or the general term of the infinite series. The sum of the first n terms of the series is denoted by s_n. That is,

$$s_n = u_1 + u_2 + u_3 + \cdots + u_n$$

A series $\sum u_n$ is said to be ***convergent*** if $\lim_{n \to \infty} s_n = l$, where l is a finite quantity and $\sum u_n$ is said to be ***divergent*** if $\lim_{n \to \infty} s_n = \pm \infty$.

Illustrative Examples

Example - 1

Let us consider the geometric series : $a + ar + ar^2 + \cdots$

$$s_n = \frac{a(1 - r^n)}{1 - r} \text{ if } r < 1 \qquad \ldots (1)$$

and $$s_n = \frac{a(r^n - 1)}{r - 1} \text{ if } r > 1 \qquad \ldots (2)$$

Now if $|r| < 1$, $r^n \to 0$ as $n \to \infty$ and from (1) we have,

$$\lim_{n \to \infty} s_n = \frac{a}{1-r}(1-0) = \frac{a}{1-r}$$ which is a finite quanitity.

Hence we conclude that the geometric series is convergent for $|r| < 1$.

Next if $r > 1$, we have from (2),

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{a(r^n - 1)}{r - 1} = \infty, \text{ since } r^n \to \infty \text{ when } r > 1.$$

Hence we conclude that the geometric series is divergent for $r > 1$.

Example - 2

Let us consider the series : $1 + 2 + 3 + \cdots + n + \cdots$

$$s_n = 1 + 2 + 3 + \cdots + n = \sum n = \frac{n(n+1)}{2}$$

$$\lim_{n \to \infty} s_n = \lim_{n \to \infty} \frac{n(n+1)}{2} = \infty$$

Hence we conclude that the series is divergent.

Example - 3

Let us consider the series : $\frac{1}{1.2} + \frac{1}{2.3} + \frac{1}{3.4} + \cdots$

The n^{th} term $u_n = \frac{1}{n(n+1)}$

Further, $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$ by partial fractions.

Now $s_n = u_1 + u_2 + u_3 + \cdots + u_n$

ie., $s_n = \left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)$

or $s_n = 1-\frac{1}{n+1}$

$\therefore \lim_{n\to\infty} s_n = \lim_{n\to\infty}\left[1-\frac{1}{n+1}\right] = 1-0 = 1$, a finite qty.

Hence we conclude that the series is convergent.

1.3 Periodic functions

A real valued function $f(x)$ is said to be ***periodic*** of ***period T*** if $f(x+T) = f(x)$, $T>0$.

k (constant), $\sin x$, $\cos x$ are periodic functions of period 2π as we know from trigonometry that

$$\sin(x+2\pi) = \sin x, \quad \cos(x+2\pi) = \cos x$$

Also if $f(x) = k$ then $f(x+2\pi) = k$

Further we also have a **property** stating that ***A linear combination of periodic functions having period T is also periodic of period T.***

1.4 Trigonometric series and Euler's formulae

The functions k, $\cos nx$, $\sin nx$ $(n = 1, 2, 3, \ldots)$ are all periodic functions of period 2π. Taking the constant $k = a_0/2$, the linear combination of all the periodic functions is of the form :

$$\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where a_n, b_n $(n = 1, 2, 3, \ldots)$ are all constants is called a **Trigonometric series.** Hence any function $f(x)$ expressible as trigonometric series of the above form must also be periodic with period 2π.

We shall assume that $f(x)$ is defined in an interval of length 2π, say $(c, c+2\pi)$ and be considered as periodic with period 2π. Then we have,

$$a_0 = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\,dx \qquad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \qquad b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx$$

The expressions for finding a_0, a_n, b_n are called ***Euler's formulae***. These can be established with the help of the following basic formulae.

1. $\int_{c}^{c+2\pi} \cos nx\, dx = 0 = \int_{c}^{c+2\pi} \sin nx\, dx$

where n is a positive integer.

2. $\int_{c}^{c+2\pi} \cos mx \cos nx\, dx = 0 = \int_{c}^{c+2\pi} \sin mx \sin nx\, dx$

where m and n are positive integers , $m \neq n$

3. $\int_{c}^{c+2\pi} \sin mx \cos nx\, dx = 0$

where m and n are positive integers.

4. $\int_{c}^{c+2\pi} \cos^2 nx\, dx = \pi = \int_{c}^{c+2\pi} \sin^2 nx\, dx$

where n is a positive integer.

Proof of Euler's formulae

Let $$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

Integrating (1) w.r.t x from c to $c + 2\pi$,

$$\int_{c}^{c+2\pi} f(x)\, dx = \int_{c}^{c+2\pi} \frac{a_0}{2}\, dx + \sum_{n=1}^{\infty} a_n \int_{c}^{c+2\pi} \cos nx\, dx + \sum_{n=1}^{\infty} b_n \int_{c}^{c+2\pi} \sin nx\, dx$$

$$= \frac{a_0}{2} \left[x \right]_{c}^{c+2\pi} + 0 + 0, \text{ by using (1)}$$

ie., $$\int_{c}^{c+2\pi} f(x)\, dx = \frac{a_0}{2} \left[c + 2\pi - c \right] = a_0 \pi$$

$\therefore$ $$a_0 = \frac{1}{\pi} \int_{c}^{c+2\pi} f(x)\, dx \qquad \ldots (2)$$

Next, taking the expanded form of (1), multiplying by $\cos nx$ and integrating w.r.t x from c to $c + 2\pi$ we have,

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = \frac{a_0}{2}\int_c^{c+2\pi}\cos nx\,dx + a_1\int_c^{c+2\pi}\cos nx\cdot\cos x\,dx$$

$$+a_2\int_c^{c+2\pi}\cos nx\,\cos 2x\,dx+\cdots+a_n\int_c^{c+2\pi}\cos^2 nx\,dx+$$

$$+b_1\int_c^{c+2\pi}\sin x\cos nx\,dx+b_2\int_c^{c+2\pi}\sin 2x\,\cos nx\,dx+\cdots$$

Using the basic results appropriately onto the R.H.S of the above we have,

$$\int_c^{c+2\pi} f(x)\cos nx\,dx = 0+0+\cdots+a_n\pi+0+0+\cdots = a_n\,\pi$$

$$\therefore \qquad a_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx\,dx \qquad \ldots(3)$$

Similarly, multiplying the expanded form of (1) by $\sin nx$ and proceeding on the same lines we obtain

$$b_n = \frac{1}{\pi}\int_c^{c+2\pi} f(x)\sin nx\,dx \qquad \ldots(4)$$

Thus we have established ***Euler's formulae.***

Remark : *The constant term in the series (1) is taken as $a_0/2$ so as to make the formula derived for a_n valid for the particular case $n = 0$ as well as for any positive integer n.*

1.5 Fourier Series of period 2π

Suppose we form the trigonometric series from $f(x)$ defined in $(c,\ c+2\pi)$ with the help of ***Euler's formulae*** we cannot conclude that the series will converge to $f(x)$. We can only say that when $f(x)$ is of the form (1) the coefficients of the terms in the series are given by the formulae (2), (3), (4).

We now proceed to state the conditions known as ***Dirichlet's conditions*** under which theexpansionof $f(x)$ asatrigonometricserieswillconvergeto $f(x)$ ateverypoint ofcontinuity.

1. $f(x)$ is single valued and finite in the interval $(c,\ c+2\pi)$
2. $f(x)$ is periodicwithperiod 2π

3. $f(x)$ has only a finite number of discontinuities in $(c,\ c+2\pi)$

4. $f(x)$ has at the most a finite number of maxima and minima in $(c,\ c+2\pi)$.

Thus we can say that, if $f(x)$ is defined in $(c,\ c+2\pi)$ and satisfies Dirichlet's conditions, then the trigonometric series (1) is called as the ***Fourier series*** of $f(x)$ in $(c,\ c+2\pi)$. The constants $a_0,\ a_n,\ b_n$ as given by (2), (3), (4) respectively are called ***Fourier coefficients.***

Remark : *If $f(x)$ is discontinuous at x, then the series (1) converges to $\frac{1}{2}\left[f(x^+)+f(x^-)\right]$ where $f(x^+), f(x^-)$ are respectively right hand and left hand limits of $f(x)$ given by*

$$f(x^+) = \lim_{h \to 0} f(x+h),\quad f(x^-) = \lim_{h \to 0} f(x-h),\quad h>0$$

However at the end points $f(x)$ converges to $\frac{1}{2}\left[f(c)+f(c+2\pi)\right]$

Note : ***Bernoulli's generalized rule of integration by parts***

While finding the Fourier coefficients, in most of the problems we have to perform integration of a product with the first function as a polynomial in x. In such cases Bernoulli's rule as given below will be highly helpful.

$$\int u\,v\,dx = u\int v\,dx - u'\iint v\,dx\,dx + u''\iiint v\,dx\,dx\,dx - \cdots$$

Here are a few illustrative examples.

1. $$\int x e^{3x}\,dx = x\left(\frac{e^{3x}}{3}\right) - (1)\left(\frac{e^{3x}}{9}\right)$$

2. $$\int (x+x^2)\cos nx\,dx$$
$$= (x+x^2)\left(\frac{\sin nx}{n}\right) - (1+2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)$$

The following integrals will be useful in problems

1. $$\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

2. $$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

The following values will have frequent reference in problems when n is an integer.

1. $\sin n\pi = 0$ **2.** $\cos n\pi = (-1)^n$

In particular $\cos(2n)\pi = +1, \cos(2n+1)\pi = -1.$

That is to say that $\cos n\pi = +1$ *when* n *is even and is equal to* -1 *when* n *is odd.*

Remark : *It is also possible to deduce a particular series from the Fourier series of a given* $f(x)$ *in a given interval. We have to substitute a suitable value for x in the given interval. Normally we take the values to be either of the end points or the middle point. The resulting series will be equal to the value as discussed in convergence.*

WORKED PROBLEMS

1. *Obtain the Fourier series of* $f(x) = \dfrac{\pi - x}{2}$ *in* $0 < x < 2\pi$. *Hence deduce that*

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\cdots = \frac{\pi}{4}$$

>> The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

where $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\, dx, a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\cos nx\, dx,\ \ b_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\sin nx\, dx$

Now, $a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \dfrac{\pi - x}{2}\, dx = \dfrac{1}{2\pi}\left[\pi x - \dfrac{x^2}{2}\right]_0^{2\pi} = \dfrac{1}{2\pi}\{(2\pi^2 - 2\pi^2) - 0\} = 0$

$\mathbf{a_0 = 0}$

$a_n = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} \dfrac{\pi - x}{2}\cos nx\, dx.$ Applying Bernoulli's rule

$$a_n = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{2\pi}$$

$= \dfrac{-1}{2\pi n^2}[\cos nx]_0^{2\pi},$ since $\sin 2n\pi = 0 = \sin 0$

$= \dfrac{-1}{2\pi n^2}\left[\cos 2n\pi - \cos 0\right] = 0,$ since $\cos 2n\pi = 1 = \cos 0$

$\mathbf{a_n = 0}$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} \frac{\pi - x}{2} \sin nx\, dx.$$ Again by Bernoulli's rule,

$$b_n = \frac{1}{2\pi}\left[(\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{2\pi}$$

$$= \frac{-1}{2\pi n}\left[(\pi - x)\cos nx\right]_0^{2\pi} + 0$$

$$= \frac{-1}{2\pi n}\left(-\pi\cos 2n\pi - \pi\cos 0\right) = \frac{-1}{2\pi n}(-\pi - \pi) = \frac{1}{n}$$

$\boldsymbol{b_n = 1/n}$

Thus by substituting the values of a_0, a_n, b_n in (1), the Fourier series is given by

$$f(x) = \frac{\pi - x}{2} = \sum_{n=1}^{\infty} \frac{1}{n}\sin nx$$

To deduce the required series we put $x = \pi/2$ in the Fourier series of $f(x)$.

[*Note that at* $x = 0$ *or* 2π, $x = \pi$ *R.H.S of the Fourier series becomes zero and hence we try* $x = \pi/2 \in (0, 2\pi)$]

$$\therefore \quad f(\pi/2) = \sum_{n=1}^{\infty} \frac{1}{n}\sin\left(\frac{n\pi}{2}\right)$$

ie., $$\frac{\pi - (\pi/2)}{2} = \sin(\pi/2) + \frac{\sin\pi}{2} + \frac{\sin(3\pi/2)}{3} + \sin 2\pi + \frac{\sin(5\pi/2)}{5} + \cdots$$

Thus $$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

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2. *Obtain the Fourier series for the function* x^2 *in* $-\pi \le x \le \pi$ *and hence deduce that*

(i) $$\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots = \frac{\pi^2}{12}$$

(ii) $$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$$

(iii) $$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}$$

>> The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

where the Fourier coefficients a_0, a_n, b_n are given by Euler's formulae.

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2\, dx$$

$$a_0 = \frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi} = \frac{1}{3\pi} \left[\pi^3 - (-\pi)^3 \right] = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$\boldsymbol{a_0/2 = \pi^2/3}$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos nx\, dx$$

$$a_n = \frac{1}{\pi} \left[(x^2) \left(\frac{\sin nx}{n} \right) - (2x) \left(\frac{-\cos nx}{n^2} \right) + (2) \left(\frac{-\sin nx}{n^3} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{2}{\pi n^2} \left[x \cos nx \right]_{-\pi}^{\pi}, \quad \text{since } \sin n\pi = 0$$

$$= \frac{2}{\pi n^2} \left[\pi \cos n\pi - (-\pi) \cos n\pi \right], \text{since } \cos(-n\pi) = \cos n\pi$$

$$= \frac{2}{\pi n^2} \cdot 2\pi \cos n\pi.$$

$$\boldsymbol{a_n = \frac{4(-1)^n}{n^2}}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx\, dx = \frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \sin nx\, dx$$

$$b_n = \frac{1}{\pi} \left[(x^2) \left(\frac{-\cos nx}{n} \right) - (2x) \left(\frac{-\sin nx}{n^2} \right) + (2) \left(\frac{\cos nx}{n^3} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left\{ \frac{-1}{n} (\pi^2 \cos n\pi - \pi^2 \cos n\pi) + 0 + \frac{2}{n^3} (\cos n\pi - \cos n\pi) \right\}$$

$$\boldsymbol{b_n = 0}$$

Thus by substituting the values of a_0, a_n, b_n in (1) the Fourier series is given by

$$f(x) = x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos nx \qquad \ldots (2)$$

Deductions : Putting $x = 0$ in (2) we get

$$f(0) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2} \cos 0$$

ie., $$0 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}, \quad \text{since} \quad f(0) = 0^2 = 0\,;\ \cos 0 = 1$$

ie., $$-\frac{\pi^2}{3} = 4\left(\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right)$$

$\therefore$ $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \qquad \ldots (1)$$

Again, putting $x = \pi$ in (2) we get

$$f(\pi) = \frac{\pi^2}{3} + \sum_{1}^{\infty} \frac{4(-1)^n}{n^2} \cos n\pi$$

ie., $$\pi^2 = \frac{\pi^2}{3} + 4\sum_{1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n, \quad \text{since } f(\pi) = \pi^2 \text{ and } \cos n\pi = (-1)^n$$

ie., $$\pi^2 - \frac{\pi^2}{3} = 4\sum_{1}^{\infty} \frac{1}{n^2} \quad \text{or} \quad \frac{2\pi^2}{3} = 4\sum_{1}^{\infty} \frac{1}{n^2}, \ \text{since } (-1)^{2n} = +1$$

$\therefore$ $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \qquad \ldots (2)$$

Now adding (1) and (2) we get,

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = 2 \cdot \frac{1}{1^2} + 2 \cdot \frac{1}{3^2} + 2 \cdot \frac{1}{5^2} + \cdots$$

ie., $$\frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

$\therefore$ $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \qquad \ldots (3)$$

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3. *If* $f(x) = x(2\pi - x)$ in $0 \le x \le 2\pi$ *show that*

$$f(x) = \frac{2\pi^2}{3} - 4\left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right)$$

Hence deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

>> The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

Now $\quad a_0 = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} f(x)\, dx = \dfrac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\, dx$

$$a_0 = \frac{1}{\pi}\left[\pi x^2 - \frac{x^3}{3}\right]_0^{2\pi} = \frac{1}{\pi}\left(4\pi^3 - \frac{8\pi^3}{3}\right) = \frac{4\pi^2}{3}$$

$$\boldsymbol{a_0/2 = 2\pi^2/3}$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\cos nx\, dx$$

$$a_n = \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{\sin nx}{n}\right) - (2\pi - 2x)\left(\frac{-\cos nx}{n^2}\right) + (-2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + \frac{1}{n^2}(2\pi - 2x)\cos nx + 0\right]_0^{2\pi} = \frac{1}{\pi n^2}(-2\pi\cos 2n\pi - 2\pi\cos 0)$$

$$\boldsymbol{a_n = \frac{-4}{n^2}} \qquad (\cos 2n\pi = 1 = \cos 0)$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\int_0^{2\pi} (2\pi x - x^2)\sin nx\, dx$$

$$b_n = \frac{1}{\pi}\left[(2\pi x - x^2)\left(\frac{-\cos nx}{n}\right) - (2\pi - 2x)\left(\frac{-\sin nx}{n^2}\right) + (-2)\left(-\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[0 + 0 + \frac{2}{n^3}\cos nx\right]_0^{2\pi}$$

[$(2\pi x - x^2)$ is zero at $x = 0,\ 2\pi$ and $\sin 2n\pi = 0 = \sin 0$]

$$b_n = \frac{2}{\pi n^3}(\cos 2n\pi - \cos 0) = \frac{2}{\pi n^3}(1 - 1) = 0$$

$$\boldsymbol{b_n = 0}$$

Thus by substituting the values of a_0, a_n, b_n in (1) we have,

$$f(x) = 2\pi x - x^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2}\cos nx$$

Expanding R.H.S by putting $n = 1, 2, 3, \ldots$ we get,

$$f(x) = \frac{2\pi^2}{3} - 4\left(\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} + \cdots\right) \qquad \ldots (2)$$

To deduce the required series we shall first put $x = 0$ in (2).

Since $f(x) = 2\pi x - x^2$ in $0 \le x \le 2\pi$, $f(0) = 0$ and hence (2) becomes

$$0 = \frac{2\pi^2}{3} - 4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right) \text{ or } \frac{-2\pi^2}{3} = -4\left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right)$$

$$\therefore \quad \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \qquad \ldots (3)$$

Again putting $x = \pi$ in (2), $f(x) = f(\pi) = 2\pi(\pi) - \pi^2 = \pi^2$ and hence (2) becomes,

$$\pi^2 = \frac{2\pi^2}{3} - 4\left(\frac{1}{1^2}\cos\pi + \frac{1}{2^2}\cos 2\pi + \frac{1}{3^2}\cos 3\pi + \cdots\right)$$

$$\pi^2 - \frac{2\pi^2}{3} = -4\left(\frac{-1}{1^2} + \frac{1}{2^2} + \frac{-1}{3^2} - \cdots\right) \text{ or } \frac{\pi^2}{3} = 4\left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots\right)$$

$$\therefore \quad \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots \qquad \ldots (4)$$

Now adding (3) and (4) we obtain,

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \cdots\right) \quad \text{or} \quad \frac{\pi^2}{4} = 2\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

Thus $\displaystyle \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

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4. *Obtain the Fourier series to represent e^{-ax} from $x = -\pi$ to $x = \pi$. Hence derive the series for $\dfrac{\pi}{\sinh \pi}$*

>> The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Now, $\displaystyle a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\, dx$

$$= \frac{1}{\pi}\left[\frac{e^{-ax}}{-a}\right]_{-\pi}^{\pi} = \frac{-1}{a\pi}\left[e^{-a\pi} - e^{a\pi}\right] = \frac{1}{a\pi}\left[e^{a\pi} - e^{-a\pi}\right]$$

$$\therefore \quad \frac{a_0}{2} = \frac{1}{a\pi}\left(\frac{e^{a\pi} - e^{-a\pi}}{2}\right) \quad \text{or} \quad \frac{a_0}{2} = \frac{\sinh a\pi}{a\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax} \cos nx\, dx.$$

We have the standard formula,

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}\,(a \cos bx + b \sin bx)$$

$$\therefore \quad a_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2 + n^2}\,(-a \cos nx + n \sin nx)\right]_{-\pi}^{\pi}$$

$$= \frac{-a}{\pi(a^2 + n^2)}\left[e^{-ax} \cos nx\right]_{-\pi}^{\pi}, \quad \text{since } \sin n\pi = 0 = \sin(-n\pi)$$

$$a_n = \frac{-a}{\pi(a^2+n^2)}\left\{e^{-a\pi}\cos n\pi - e^{a\pi}\cdot\cos(-n\pi)\right\}$$

$$= \frac{-a\cos n\pi}{\pi(a^2+n^2)}(e^{-a\pi}-e^{a\pi}) = \frac{a(-1)^n}{\pi(a^2+n^2)}(e^{a\pi}-e^{-a\pi})$$

$$\boldsymbol{a_n = \frac{2a(-1)^n \sinh a\pi}{\pi(a^2+n^2)}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} e^{-ax}\sin nx\,dx.$$

We have the standard formula,

$$\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$$

$$\therefore \quad b_n = \frac{1}{\pi}\left[\frac{e^{-ax}}{(-a)^2+n^2}(-a\sin nx - n\cos nx)\right]_{-\pi}^{\pi}$$

$$= \frac{-n}{\pi(a^2+n^2)}\left[e^{-ax}\cos nx\right]_{-\pi}^{\pi}$$

The function to be evaluated between the limits $-\pi$ to π is same as in a_n

$$\boldsymbol{b_n = \frac{2n(-1)^n \sinh a\pi}{\pi(a^2+n^2)}}$$

Substituting the values of a_0, a_n, b_n in (1) the Fourier series is given by

$$f(x) = \frac{\sinh a\pi}{a\pi} + \sum_{n=1}^{\infty}\frac{2a(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\cos nx + \sum_{n=1}^{\infty}\frac{2n(-1)^n\sinh a\pi}{\pi(a^2+n^2)}\sin nx$$

Thus $$\boldsymbol{e^{-ax} = \frac{\sinh a\pi}{a\pi}\left\{1 + \sum_{1}^{\infty}\frac{2a^2(-1)^n}{a^2+n^2}\cos nx + \sum_{1}^{\infty}\frac{2an(-1)^n}{a^2+n^2}\sin nx\right\}}$$

To deduce the series we shall put $a = 1, x = 0$ in the Fourier series.

$$e^0 = \frac{\sinh\pi}{\pi}\left\{1 + \sum_{1}^{\infty}\frac{2(-1)^n}{1+n^2}\right\} \quad \text{since } \cos 0 = 1,\ \sin 0 = 0$$

ie., $$1 = \frac{\sinh\pi}{\pi}\left\{1 + 2\sum_{1}^{\infty}\frac{(-1)^n}{1+n^2}\right\} \quad \text{or} \quad \frac{\pi}{\sinh\pi} = 1 + 2\sum_{1}^{\infty}\frac{(-1)^n}{1+n^2}$$

ie., $$\frac{\pi}{\sinh \pi} = 1 + 2\left(\frac{-1}{1+1^2} + \frac{1}{1+2^2} - \frac{1}{1+3^2} + \frac{1}{1+4^2} - \cdots\right)$$

$$= 1 - 1 + 2\left(\frac{1}{5} - \frac{1}{10} + \frac{1}{17} - \cdots\right)$$

Thus $$\frac{\pi}{\sin h\pi} = 2\left(\frac{1}{5} - \frac{1}{10} + \frac{1}{17} - \cdots\right)$$

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5. *Find a Fourier series in* $(-\pi, \pi)$ *to represent the following functions .*

(a) $f(x) = x - x^2$ **(b)** $f(x) = x + x^2$

Hence deduce that

(i) $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\cdots$$

(ii) $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots\cdots$$

(iii) $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots\cdots$$

>> **(a)** Period of $f(x) = \pi - (-\pi) = 2\pi$ and the Fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \text{ where}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx,\ a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx,\ b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} (x - x^2)\,dx = \frac{1}{\pi}\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left\{\left(\frac{\pi^2}{2} - \frac{\pi^2}{2}\right) - \left(\frac{\pi^3}{3} - \frac{-\pi^3}{3}\right)\right\} = \frac{-2\pi^2}{3}$$

$$a_0/2 = -\pi^2/3$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \cos n\,x\,dx$$

$$= \frac{1}{\pi} \left[(x - x^2) \left(\frac{\sin nx}{n} \right) - (1 - 2x) \left(\frac{-\cos nx}{n^2} \right) + (-2) \left(\frac{-\sin nx}{n^3} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi n^2} [(1 - 2x) \cos nx]_{-\pi}^{\pi} = \frac{1}{\pi n^2} \quad (1 - 2\pi) \cos n\pi - (1 + 2\pi) \cos n\pi \,;$$

$$\therefore \qquad a_n = \frac{-4}{n^2} \cos n\pi = -\frac{4}{n^2} (-1)^n$$

$$a_n = \frac{4(-1)^{n+1}}{n^2}$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} (x - x^2) \sin nx\,dx$$

$$= \frac{1}{\pi} \left[(x - x^2) \left(\frac{-\cos nx}{n} \right) - (1 - 2x) \left(\frac{-\sin nx}{n^2} \right) + (-2) \left(\frac{\cos nx}{n^3} \right) \right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[\frac{-1}{n} \left\{ (\pi - \pi^2) \cos n\pi - (-\pi - \pi^2) \cos n\pi - \frac{2}{n^3} (\cos n\pi - \cos n\pi) \right\} \right]$$

$$b_n = -\frac{1}{\pi n} (2\pi \cos n\pi) = \frac{-2}{n} (-1)^n$$

$$b_n = \frac{2}{n} (-1)^{n+1}$$

The required Fourier series is given by

$$x - x^2 = \frac{-\pi^2}{3} + 4 \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx + 2 \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx \qquad \ldots (1)$$

To deduce the series, first let us put $x = 0$ which is a point of the interval $(-\pi, \pi)$. Hence (1) becomes

$$0 = -\frac{\pi^2}{3} + 4 \sum_{1}^{\infty} \frac{(-1)^{n+1}}{n^2} \cdot 1 + 0$$

ie., $$\frac{\pi^2}{3} = 4\left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right]$$

or $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \qquad \ldots (1)$$

Next let us put $x = \pi$ in (1). Since $f(x) = x - x^2$ is defined in $-\pi < x < \pi$ the value of $f(x)$ at $x = \pi$ being $f(\pi)$ is given by $\frac{1}{2}\,[f(-\pi)+f(\pi)]$ which being $\frac{1}{2}\left[(-\pi-\pi^2)+(\pi-\pi^2)\right] = -\pi^2$

Hence (1) becomes

$$-\pi^2 = \frac{-\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^{n+1}}{n^2}\cos n\pi + 0$$

ie., $$-\pi^2 + \frac{\pi^2}{3} = 4\sum_1^\infty \frac{(-1)^{2n+1}}{n^2}, \quad \text{since } \cos n\pi = (-1)^n$$

ie., $$-\frac{\pi^2}{6} = \frac{-1}{1^2} - \frac{1}{2^2} - \frac{1}{3^2} - \frac{1}{4^2} - \cdots$$

or $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \qquad \ldots (2)$$

Adding (1) and (2) we obtain

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \qquad \ldots (3)$$

(b) Fourier series of $f(x) = x + x^2$ in $-\pi < x < \pi$ on similar lines can be obtained in the form

$$x + x^2 = \frac{\pi^2}{3} + 4\sum_1^\infty \frac{(-1)^n}{n^2}\cos nx + 2\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin nx$$

All the three series can be deduced in a similar way.

6. *If* $f(x) = \begin{cases} x & in\ \ 0 \le x \le \pi \\ 2\pi - x & in\ \ \pi \le x \le 2\pi \end{cases}$ *show that the Fourier series of* $f(x)$ *in* $[0,\ 2\pi]$ *is* $\frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$ *and hence deduce that* $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

>> The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

Now, $a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\, dx + \int_{\pi}^{2\pi} f(x)\, dx\right\}$

ie., $a_0 = \frac{1}{\pi}\left\{\int_0^{\pi} x\, dx + \int_{\pi}^{2\pi} (2\pi - x)\, dx\right\}$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_0^{\pi} + \left[2\pi x - \frac{x^2}{2}\right]_{\pi}^{2\pi}\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\left(\frac{\pi^2}{2} - 0\right) + (4\pi^2 - 2\pi^2) - \left(2\pi^2 - \frac{\pi^2}{2}\right)\right\} = \frac{1}{\pi}(\pi^2) = \pi$$

$$a_0/2 = \pi/2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\cos nx\, dx + \int_{\pi}^{2\pi} f(x)\cos nx\, dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\int_0^{\pi} x\cos nx\, dx + \int_{\pi}^{2\pi} (2\pi - x)\cos nx\, dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{\sin nx}{n}\right) - (-1)\left(\frac{-\cos nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

where we have applied Bernoulli's rule to each of the integrals.

$$a_n = \frac{1}{\pi n^2}\left\{ [\cos nx]_0^{\pi} - [\cos nx]_{\pi}^{2\pi} \right\}, \quad \text{since } \sin n\pi = 0 = \sin 0$$

$$= \frac{1}{\pi n^2}\{(\cos n\pi - 1) - (1 - \cos n\pi)\}, \quad \text{since } \cos 0 = 1 = \cos 2n\pi$$

$$= \frac{1}{\pi n^2}(-2 + 2\cos n\pi) = \frac{-2}{\pi n^2}(1 - \cos n\pi)$$

$$\boldsymbol{a_n = \frac{-2}{\pi n^2}\{1 - (-1)^n\}}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_0^{\pi} f(x)\sin nx\,dx + \int_{\pi}^{2\pi} f(x)\sin nx\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi} + \left[(2\pi - x)\left(\frac{-\cos nx}{n}\right) - (-1)\left(\frac{-\sin nx}{n^2}\right)\right]_{\pi}^{2\pi}\right\}$$

$$= \frac{-1}{\pi n}\left\{[x\cos nx]_0^{\pi} + 0 + [(2\pi - x)\cos nx]_{\pi}^{2\pi} - 0\right\}$$

$$b_n = \frac{-1}{\pi n}\{(\pi\cos n\pi - 0) + (0 - \pi\cos n\pi)\} = 0$$

$$\boldsymbol{b_n = 0}$$

The Fourier series representation of $f(x)$ is given by

$$\boldsymbol{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty}\frac{-2}{\pi n^2}\{1 - (-1)^n\}\cos nx}$$

But $\quad 1 - (-1)^n = \begin{cases} 1 - (-1) = 2 \text{ if } n \text{ is odd} \\ 1 - (+1) = 0 \text{ if } n \text{ is even} \end{cases}$

$$\therefore \quad f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1,3,5\ldots}^{\infty}\frac{1}{n^2}\cdot 2\cos nx$$

Thus $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \cdots\right]$

To deduce the series let us put $x = 0$.

Then $f(x) = 0$ since $f(x) = x$ in $0 \le x \le \pi$.

Hence the Fourier series becomes

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right] \quad \text{or} \quad \frac{-\pi}{2} = \frac{-4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

Thus $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

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7. *Obtain the Fourier series for the function*

$$f(x) = \begin{cases} -\pi & in \ -\pi < x < 0 \\ x & in \ 0 < x < \pi \end{cases}$$ *Hence deduce that,*

the sum of the reciprocal squares of the odd integers is equal to $\pi^2/8$.

>> $f(x)$ is defined in $(-\pi, \pi)$ and the Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\,dx + \int_{0}^{\pi} x\,dx\right\} = \frac{1}{\pi}\left\{-\pi\,[x]_{-\pi}^{0} + \left[\frac{x^2}{2}\right]_0^{\pi}\right\}$$

$$a_0 = \frac{1}{\pi}\left\{-\pi\left(0-(-\pi)\right) + \left(\frac{\pi^2}{2} - 0\right)\right\}$$

$$a_0 = \frac{1}{\pi}\left(-\pi^2 + \frac{\pi^2}{2}\right) = \frac{1}{\pi}\left(\frac{-\pi^2}{2}\right) = -\frac{\pi}{2}$$

$$a_0/2 = -\pi/4$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\cos nx\,dx + \int_{0}^{\pi} x\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\pi\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[x\cdot\frac{\sin nx}{n}\right]_{0}^{\pi} - \left[1\cdot\frac{-\cos nx}{n^2}\right]_{0}^{\pi}\right\}$$

where the integration is carried out by Bernoulli's rule in the second integral.

$$a_n = \frac{1}{\pi n^2}\,[\cos nx]_0^{\pi} = \frac{1}{\pi n^2}\,(\cos n\pi - \cos 0), \text{ since } \sin n\pi = 0 = \sin 0$$

$$\mathbf{a_n = \frac{1}{\pi n^2}\left\{(-1)^n - 1\right\} \quad or \quad a_n = \frac{-1}{\pi n^2}\left\{1-(-1)^n\right\}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} -\pi\sin nx\,dx + \int_{0}^{\pi} x\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{-\pi\left[\frac{-\cos nx}{n}\right]_{-\pi}^{0} + \left[x\cdot\frac{-\cos nx}{n}\right]_{0}^{\pi} - \left[1\cdot\frac{-\sin nx}{n^2}\right]_{0}^{\pi}\right\}$$

$$= \frac{1}{\pi n}\left\{\pi[\cos nx]_{-\pi}^{0} - [x\cos nx]_{0}^{\pi}\right\}, \quad \text{since } \sin n\pi = 0 = \sin 0$$

$$= \frac{1}{\pi n}\left\{\pi(\cos 0 - \cos n\pi) - (\pi\cos n\pi - 0)\right\}$$

$$b_n = \frac{\pi}{\pi n}\left\{1-\cos n\pi - \cos n\pi\right\} = \frac{1}{n}\,(1-2\cos n\pi)$$

$$b_n = \frac{1}{n}\left\{1-2(-1)^n\right\}$$

Substituting the values of a_0, a_n, b_n in (1), the Fourier series is given by

$$f(x) = \frac{-\pi}{4} + \sum_{n=1}^{\infty} \frac{-1}{\pi n^2}\left\{1-(-1)^n\right\}\cos nx + \sum_{n=1}^{\infty}\frac{1}{n}\left\{1-2(-1)^n\right\}\sin nx$$

To deduce the required series let us put $x = 0$ in the Fourier series.

It should be observed from the given $f(x)$ that $x = 0$ is a point of discontinuity and hence the series converges to

$$\frac{1}{2}\left[f(0^+)+f(0^-)\right] = \frac{1}{2}\left[0+(-\pi)\right] = \frac{-\pi}{2}$$

Because to the right of 0, in $(0, \pi), f(x) = x$ and $f(0^+) = 0$. Also to the left of 0, in $(-\pi, 0), f(x) = -\pi$ and $f(0^-) = -\pi$

Hence the Fourier series becomes

$$-\frac{\pi}{2} = -\frac{\pi}{4} + \sum_{n=1}^{\infty}\frac{-1}{\pi n^2}\left\{1-(-1)^n\right\}, \quad \text{since} \quad \cos 0 = 1, \sin 0 = 0$$

ie., $$-\frac{\pi}{2}+\frac{\pi}{4} = -\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{\pi^2}\left\{1-(-1)^n\right\}$$

ie., $$-\frac{\pi}{4} = -\frac{1}{\pi}\sum_{1}^{\infty}\frac{1}{n^2}\left\{1-(-1)^n\right\} \quad \text{or} \quad \frac{\pi^2}{4} = \sum_{1}^{\infty}\frac{1}{n^2}\left\{1-(-1)^n\right\}$$

But $$1-(-1)^n = \begin{cases} 1-(+1) = 0 & \text{when } n \text{ is even} \\ 1-(-1) = 2 & \text{when } n \text{ is odd}\end{cases}$$

Hence we get $$\frac{\pi^2}{4} = \sum_{n=1,3,5,\cdots}^{\infty}\frac{1}{n^2}(2) \quad \text{or} \quad \frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots$$

Thus the sum of the reciprocal squares of the odd integers is $\pi^2/8$.

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8. *Obtain the Fourier series for the function*

$$f(x) = \begin{cases} 0 & in\ -\pi \le x \le 0 \\ \sin x & in\ \ 0 \le x \le \pi \end{cases} \qquad \textit{Deduce that}$$

1. $\frac{1}{1.3}+\frac{1}{3.5}+\frac{1}{5.7}+\cdots = \frac{1}{2}$ **2.** $\frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots = \frac{\pi-2}{4}$

>> $f(x)$ is defined in $[-\pi, \pi]$ and the Fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0\, dx + \int_{0}^{\pi} \sin x\, dx\right\} = \frac{1}{\pi}\int_{0}^{\pi} \sin x\, dx$$

$$a_0 = \frac{1}{\pi}\Big[-\cos x\Big]_0^{\pi} = \frac{-1}{\pi}(\cos\pi - \cos 0) = -\frac{1}{\pi}(-1-1) = \frac{2}{\pi}$$

$$\mathbf{a_0/2 = 1/\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0 \cdot \cos nx\, dx + \int_{0}^{\pi} \sin x \cdot \cos nx\, dx\right\}$$

ie., $$a_n = \frac{1}{\pi}\int_{0}^{\pi} \sin x \cos nx\, dx.$$

Using $\sin A \cos B = \frac{1}{2}\{\sin(A+B) + \sin(A-B)\}$

$$a_n = \frac{1}{\pi}\int_{0}^{\pi} \frac{1}{2}\{\sin(x+nx) + \sin(x-nx)\}\, dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\{\sin(1+n)x + \sin(1-n)x\}\, dx$$

$$= \frac{1}{2\pi}\int_{0}^{\pi}\{\sin(n+1)x - \sin(n-1)x\}\, dx$$

$$= \frac{1}{2\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi} \quad \text{where } n \neq 1$$

$$= \frac{1}{2\pi}\left[\frac{-1}{n+1}\{\cos(n+1)\pi - \cos 0\} + \frac{1}{n-1}\{\cos(n-1)\pi - \cos 0\}\right]$$

Using $\cos 0 = 1, \cos k\pi = (-1)^k$ and rearranging ,

$$a_n = \frac{1}{2\pi}\left\{\left(\frac{1}{n+1} - \frac{1}{n-1}\right) + \left(\frac{-1}{n+1}(-1)^{n+1} + \frac{1}{n-1}(-1)^{n-1}\right)\right\}$$

$$a_n = \frac{1}{2\pi}\left\{\frac{-2}{n^2-1} + \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-2}{n^2-1} + (-1)^n\left(\frac{(-1)^2}{n+1} + \frac{(-1)^{-1}}{n-1}\right)\right\}$$

$$= \frac{1}{2\pi}\left\{\frac{-2}{n^2-1} + (-1)^n\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\right\} \quad \because (-1)^{-1} = 1/-1 = -1$$

$$= \frac{1}{2\pi}\left\{\frac{-2}{n^2-1} + (-1)^n\frac{-2}{n^2-1}\right\}$$

$$\mathbf{a_n = \frac{-1}{\pi(n^2-1)}\{1+(-1)^n\}} \quad \text{where } n \neq 1$$

Now we shall find a_n when $n = 1$. That is to find a_1

We have $a_n = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin x \cos nx \, dx$

Putting $n = 1, a_1 = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin x \cos x \, dx = \dfrac{1}{\pi}\int_0^{\pi} \dfrac{\sin 2x}{2}\, dx$

ie., $a_1 = \dfrac{1}{2\pi}\left[\dfrac{-\cos 2x}{2}\right]_0^{\pi} = \dfrac{-1}{4\pi}(\cos 2\pi - \cos 0) = \dfrac{-1}{4\pi}(1-1) = 0$

$\therefore$ $\mathbf{a_1 = 0}$ **and** $\mathbf{a_n = \dfrac{-1}{\pi(n^2-1)}\{1+(-1)^n\}}$, for $n \neq 1$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} 0 \cdot \sin nx\, dx + \int_0^{\pi} \sin x \sin nx\, dx\right\}$$

ie., $b_n = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \sin x \sin nx\, dx$

Using $\sin A \sin B = \dfrac{1}{2}\{\cos(A-B) - \cos(A+B)\}$,

$$b_n = \frac{1}{\pi} \int_0^{\pi} \frac{1}{2} \{ \cos(x - nx) - \cos(x + nx) \} \, dx$$

$$= \frac{1}{2\pi} \int_0^{\pi} \{ \cos(1 - n)x - \cos(1 + n)x \} \, dx$$

$$= \frac{1}{2\pi} \int_0^{\pi} \{ \cos(n - 1)x - \cos(n + 1)x \} \, dx, \quad \text{since } \cos(-\theta) = \cos\theta$$

$$= \frac{1}{2\pi} \left[\frac{\sin(n-1)x}{n-1} - \frac{\sin(n+1)x}{n+1} \right]_0^{\pi} \quad \text{where } n \neq 1$$

$\boldsymbol{b_n = 0 \ (n \neq 1)}$ since $\sin k\pi = 0$ for integral values of k.

Now we shall find b_n when $n = 1$. That is to find b_1.

We have $b_n = \dfrac{1}{\pi} \displaystyle\int_0^{\pi} \sin x \, \sin nx \, dx$

Putting $n = 1$, $b_1 = \dfrac{1}{\pi} \displaystyle\int_0^{\pi} \sin x \cdot \sin x \, dx = \dfrac{1}{\pi} \displaystyle\int_0^{\pi} \sin^2 x \, dx$

ie., $$b_1 = \frac{1}{\pi} \int_0^{\pi} \frac{1}{2} (1 - \cos 2x) \, dx = \frac{1}{2\pi} \left[x - \frac{\sin 2x}{2} \right]_0^{\pi}$$

$$= \frac{1}{2\pi} (\pi - 0) = \frac{1}{2}, \quad \text{since } \sin 2\pi = 0 = \sin 0$$

$\therefore \quad \boldsymbol{b_1 = 1/2, \ b_n = 0 \ \text{for} \ n \neq 1}$

Let us consider (1) in the form,

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

Thus by substituting the values of the Fourier coefficients, the required Fourier series is given by

$$\boldsymbol{f(x) = \frac{1}{\pi} + \sum_{n=2}^{\infty} \frac{-1}{\pi(n^2 - 1)} \{ 1 + (-1)^n \} \cos nx + \frac{1}{2} \sin x}$$

To deduce the required series we shall first put $x = 0$

$\therefore \quad f(0) = \dfrac{1}{\pi} + \sum_{2}^{\infty} \dfrac{-1}{\pi(n^2-1)} \{1+(-1)^n\} \cdot 1 + 0$

Also $\quad f(0) = 0$ as we have $f(x) = 0$ in $-\pi \le x \le 0$

ie., $\quad 0 = \dfrac{1}{\pi} + \sum_{2}^{\infty} \dfrac{-1}{\pi(n^2-1)} \{1+(-1)^n\}$

ie., $\quad \dfrac{-1}{\pi} = \dfrac{-1}{\pi} \sum_{2}^{\infty} \dfrac{-1}{n^2-1} \{1+(-1)^n\}$

ie., $\quad 1 = \sum_{2}^{\infty} \dfrac{1}{n^2-1} \{1+(-1)^n\}$

But $\quad 1+(-1)^n = \begin{cases} 1+1 = 2 \text{ when } n \text{ is even} \\ 1-1 = 0 \text{ when } n \text{ is odd} \end{cases}$

$\therefore \quad 1 = \sum_{n=2,4,6,\cdots}^{\infty} \dfrac{1}{n^2-1} (2)$ or $\dfrac{1}{2} = \sum_{n=2,4,\cdots}^{\infty} \dfrac{1}{n^2-1}$

Thus $\quad \mathbf{\dfrac{1}{2} = \dfrac{1}{3} + \dfrac{1}{15} + \dfrac{1}{35} + \cdots}$ or $\mathbf{\dfrac{1}{2} = \dfrac{1}{1.3} + \dfrac{1}{3.5} + \dfrac{1}{5.7} + \cdots}$

Next let us put $x = \pi/2$ in the Fourier series.

$\therefore \quad f\left(\dfrac{\pi}{2}\right) = \dfrac{1}{\pi} + \sum_{n=2}^{\infty} \dfrac{-1}{\pi(n^2-1)} \{1+(-1)^n\} \cos \dfrac{n\pi}{2} + \dfrac{1}{2} \cdot 1$

Also $\quad f(\pi/2) = \sin(\pi/2) = 1, \quad$ since $f(x) = \sin x$ in $0 \le x \le \pi$

ie., $\quad 1 = \dfrac{1}{\pi} + \sum_{n=2}^{\infty} \dfrac{-1}{\pi(n^2-1)} \{1+(-1)^n\} \cos \dfrac{n\pi}{2} + \dfrac{1}{2}$

ie., $\quad 1 - \dfrac{1}{2} - \dfrac{1}{\pi} = \dfrac{-1}{\pi} \sum_{n=2}^{\infty} \dfrac{1}{n^2-1} \{1+(-1)^n\} \cos \dfrac{n\pi}{2}$

ie., $\quad \dfrac{\pi-2}{2\pi} = \dfrac{-1}{\pi} \sum_{n=2}^{\infty} \dfrac{1}{n^2-1} \{1+(-1)^n\} \cos \dfrac{n\pi}{2}$

ie., $\quad \dfrac{\pi-2}{2} = -\sum_{n=2,4,6,\cdots}^{\infty} \dfrac{1}{n^2-1} (2) \cos \dfrac{n\pi}{2}$

ie., $\quad \dfrac{\pi-2}{4} = -\left\{\dfrac{1}{3}\cos\pi + \dfrac{1}{15}\cos 2\pi + \dfrac{1}{35}\cos 3\pi + \cdots\right\}$

But $\quad \cos\pi = -1 = \cos 3\pi, \; \cos 2\pi = 1$

$$\therefore \quad \frac{\pi-2}{4} = -\left(-\frac{1}{3}+\frac{1}{15}-\frac{1}{35}+\cdots\right) = \frac{1}{3}-\frac{1}{15}+\frac{1}{35}-\cdots$$

Thus $$\frac{\pi-2}{4} = \frac{1}{1.3}-\frac{1}{3.5}+\frac{1}{5.7}-\cdots$$

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9. *An alternating current after passing through a rectifier has the form*

$$I = \begin{cases} I_0 \sin\theta & \text{for } 0 < \theta \le \pi \\ 0 & \text{for } \pi < \theta \le 2\pi \end{cases}$$

where I_0 is the maximum current. Express I as a Fourier series in $(0,\ 2\pi)$

>> The Fourier series of period of 2π is given by

$$I = f(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta \qquad \ldots (1)$$

We have to find the Fourier coefficients by using Euler's formulae for the interval $(0,\ 2\pi)$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\,d\theta,\quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta\,d\theta,\quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta\,d\theta$$

Each of the above integrals after splitting into two integrals in the range $(0,\ \pi)$, $(\pi,\ 2\pi)$ and substituting for $f(\theta)$ will give us

$$a_0 = \frac{I_0}{\pi}\int_0^{\pi} \sin\theta\,d\theta,\quad a_n = \frac{I_0}{\pi}\int_0^{\pi} \sin\theta\cos n\theta\,d\theta,\quad b_n = \frac{I_0}{\pi}\int_0^{\pi} \sin\theta\sin n\theta\,d\theta$$

These integrals are same as in the previous problem and hence the Fourier coefficients are as follows.

$$a_0 = \frac{2I_0}{\pi},\quad a_n = \frac{-I_0}{\pi(n^2-1)}\left\{1+(-1)^n\right\} \text{ if } n \ne 1, a_1 = 0$$

$$b_n = 0 \text{ if } n \ne 1 \text{ and } b_1 = \frac{I_0}{2}$$

Thus by substituting these values in (1) the required Fourier series is given by

$$I = f(\theta) = \frac{I_0}{\pi} + \sum_{n=2}^{\infty} \frac{-1}{\pi(n^2-1)}\left\{1+(-1)^n\right\}\cos n\theta + \frac{I_0}{2}\sin\theta$$

---

10. *Obtain the Fourier series for*

$$f(x) = \begin{cases} -k \text{ in } (-\pi, 0) \\ +k \text{ in } (0, \pi) \end{cases} \quad \textit{Hence deduce that } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$$

>> The Fourier series of $f(x)$ defined in $(-\pi, \pi)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\, dx + \int_{0}^{\pi} f(x)\, dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\, dx + \int_{0}^{\pi} k\, dx\right\} = \frac{k}{\pi}\left\{\left[-x\right]_{-\pi}^{0} + \left[x\right]_{0}^{\pi}\right\}$$

$$a_0 = \frac{k}{\pi}\{-(0+\pi)+(\pi-0)\} = 0$$

$$\boldsymbol{a_0/2 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\, dx + \int_{0}^{\pi} f(x)\cos nx\, dx\right\}$$

$$a_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\cos nx\, dx + \int_{0}^{\pi} k\cos nx\, dx\right\}$$

$$= \frac{k}{\pi}\left\{-\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left[\frac{\sin nx}{n}\right]_{0}^{\pi}\right\} = 0, \quad \text{since } \sin 0 = 0 = \sin n\pi$$

$$\boldsymbol{a_n = 0}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\, dx + \int_{0}^{\pi} f(x)\sin nx\, dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0} -k\sin nx\, dx + \int_{0}^{\pi} k\sin nx\, dx\right\}$$

$$b_n = \frac{k}{\pi}\left\{\left[\frac{\cos nx}{n}\right]_{-\pi}^{0} + \left[-\frac{\cos nx}{n}\right]_{0}^{\pi}\right\}$$

$$= \frac{k}{\pi n}\{(1-\cos n\pi)-(\cos n\pi - 1)\} = \frac{k}{\pi n}(2-2\cos n\pi)$$

$$\mathbf{b_n = \frac{2k}{\pi n}\{1-(-1)^n\}}$$

Thus the required Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{2k}{\pi n}\{1-(-1)^n\} \sin nx$$

But $\quad 1-(-1)^n = \begin{cases} 1-(+1) = 0 \text{ if } n \text{ is even} \\ 1-(-1) = 2 \text{ if } n \text{ is odd} \end{cases}$

$$\therefore \quad f(x) = \frac{4k}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n} \sin nx$$

$$f(x) = \frac{4k}{\pi}\left\{\frac{\sin x}{1} + \frac{\sin 2x}{2} + \frac{\sin 3x}{3} + \frac{\sin 4x}{4} + \cdots\right\}$$

To deduce the series let us put $x = \pi/2$.

Then $f(x) = k$ since $f(x) = k$ in $0 < x < \pi$.

Hence the Fourier series becomes

$$k = \frac{4k}{\pi}\left(1 - \frac{1}{3} + \frac{1}{5} - \cdots\right), \text{ since } \sin\left(\frac{3\pi}{2}\right) = -1, \sin\left(\frac{5\pi}{2}\right) = 1$$

Thus $\quad \mathbf{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots}$

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11. *Find the Fourier series of*

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi} & in\ -\pi < x \le 0 \\ 1 - \frac{2x}{\pi} & in\ 0 \le x < \pi \end{cases} \qquad \textit{Hence deduce that } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

\>> The Fourier series of $f(x)$ defined in $(-\pi, \pi)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\,dx + \int_{0}^{\pi} f(x)\,dx\right\}$$

$$a_0 = \frac{1}{\pi}\left\{\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)dx + \int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[x+\frac{x^2}{\pi}\right]_{-\pi}^{0} + \left[x-\frac{x^2}{\pi}\right]_{0}^{\pi}\right\}$$

$$a_0 = \frac{1}{\pi}\{0-(-\pi+\pi)+(\pi-\pi)-0\} = 0$$

$$\mathbf{a_0/2 = 0}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\cos nx\,dx + \int_{0}^{\pi} f(x)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)\cos nx\,dx + \int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\left(1+\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_{-\pi}^{0}\right.$$

$$\left. + \left[\left(1-\frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

But $\sin n\pi = 0 = \sin 0$

$$a_n = \frac{2}{\pi^2 n^2}\{(1-\cos n\pi)-(\cos n\pi - 1)\} = \frac{2}{\pi^2 n^2}(2-2\cos n\pi)$$

$$\mathbf{a_n = \frac{4}{\pi^2 n^2}\{1-(-1)^n\}}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} f(x)\sin nx\,dx + \int_{0}^{\pi} f(x)\sin nx\,dx\right\}$$

$$b_n = \frac{1}{\pi}\left\{\int_{-\pi}^{0}\left(1+\frac{2x}{\pi}\right)\sin nx\,dx + \int_{0}^{\pi}\left(1-\frac{2x}{\pi}\right)\sin nx\,dx\right\}$$

$$= \frac{1}{\pi}\left\{\left[\left(1+\frac{2x}{\pi}\right)\left(\frac{-\cos nx}{n}\right)-\left(\frac{2}{\pi}\right)\left(\frac{-\sin nx}{n^2}\right)\right]_{-\pi}^{0}\right.$$

$$\left. + \left[\left(1-\frac{2x}{\pi}\right)\left(\frac{-\cos nx}{n}\right)-\left(\frac{-2}{\pi}\right)\left(\frac{-\sin nx}{n^2}\right)\right]_{0}^{\pi}\right\}$$

$$= \frac{-1}{\pi n}\left\{\left[\left(1+\frac{2x}{\pi}\right)(\cos nx)\right]_{-\pi}^{0} + \left[\left(1-\frac{2x}{\pi}\right)(\cos nx)\right]_{0}^{\pi}\right\}$$

$$= \frac{-1}{\pi n}\left[1+\cos n\pi - \cos n\pi - 1\right] = 0$$

$$\boldsymbol{b_n = 0}$$

Thus the required Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]\cos nx$$

But $\quad 1-(-1)^n = \begin{cases} 1-(+1) = 0 \text{ if } n \text{ is even} \\ 1-(-1) = 2 \text{ if } n \text{ is odd} \end{cases}$

$$\therefore \quad f(x) = \frac{8}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty} \frac{\cos nx}{n^2}$$

or $\quad f(x) = \dfrac{8}{\pi^2}\left(\dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \dfrac{\cos 5x}{5^2} + \cdots\right)$

Now putting $x = 0, f(x) = 1$. The Fourier series becomes

$$1 = \frac{8}{\pi^2}\left(\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\right)$$

Thus $\quad \dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

- -

1.6 Fourier series of even and odd functions

Definition : A function $f(x)$ is said to be **even** if $f(-x) = f(x)$ and **odd** if $f(-x) = -f(x)$

For example, $x^2, x^4, x^6, \ldots \cos x$ are even functions and $x, x^3, x^5, \ldots \sin x$ are odd functions.

Property-1 : The product of two even functions and that of two odd functions is always even whereas the product of an even and an odd function is always odd.

Property-2 : $$\int_{-a}^{+a} \phi(x)\, dx = \begin{cases} 2\int_0^a \phi(x)\, dx & \text{if } \phi(x) \text{ is even} \\ 0 & \text{if } \phi(x) \text{ is odd} \end{cases}$$

Now, suppose the periodic function $f(x)$ is defined in the interval $(-\pi, \pi)$ then the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx, \quad a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx\, dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx\, dx$$

Let us examine the following two cases.

Case-i : $f(x)$ is an even function

$f(x) \cos nx$ being the products of two even functions is also even and $f(x) \sin nx$ being the product of an even and odd function is odd [*Property*-1]. Now applying property-2 to these integrals we have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx, \; b_n = 0$$

Case-ii : $f(x)$ is an odd function

$f(x) \cos nx$ will be an odd function and $f(x) \sin nx$ will be an even function. Now by property-2 we have

$$a_0 = 0, \quad a_n = 0, \quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx$$

Thus we can conclude that when $x \in (-\pi, \pi)$, if $f(x)$ is even $b_n = 0$ and if $f(x)$ is odd $a_0 = 0,\ a_n = 0$

Note : *If* $f(x) = \begin{cases} \phi(x) & in \ -\pi < x < 0 \\ \psi(x) & in \ 0 < x < \pi \end{cases}$

we say that $f(x)$ *is even if* $\phi(-x) = \psi(x)$ *and* $f(x)$ *is odd if* $\phi(-x) = -\psi(x)$

Examples

1. $f(x) = \begin{cases} -x & \text{in } -\pi < x < 0 \\ +x & \text{in } 0 < x < \pi \end{cases}$ is an even function

2. $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} & \text{in } -\pi < x < 0 \\ 1 - \dfrac{2x}{\pi} & \text{in } 0 < x < \pi \end{cases}$ is an even function

3. $f(x) = \begin{cases} x - \dfrac{\pi}{2} & \text{in } -\pi < x < 0 \\ x + \dfrac{\pi}{2} & \text{in } 0 < x < \pi \end{cases}$ is an odd function because

   if $\phi(x) = x - \dfrac{\pi}{2}$ then $\phi(-x) = -x - \dfrac{\pi}{2} = -\left(x + \dfrac{\pi}{2}\right) = -\psi(x)$

4. $f(x) = \begin{cases} -k & \text{in } -\pi < x < 0 \\ +k & \text{in } 0 < x < \pi \end{cases}$ is an odd function because

   if $\phi(x) = -k$, then $\phi(-x) = -k = -\psi(x)$

5. $f(x) = |x|$ is an even function, since $|-x| = |x|$

The results are tabulated for ready reference where $f(x)$ is defined in $(-\pi, \pi)$

Nature & condition of $f(x)$	a_0	a_n	b_n
Even function $f(-x) = f(x)$	$\frac{2}{\pi}\int_0^\pi f(x)\,dx$	$\frac{2}{\pi}\int_0^\pi f(x)\cos nx\,dx$	0
Odd function $f(-x) = -f(x)$	0	0	$\frac{2}{\pi}\int_0^\pi f(x)\sin nx\,dx$

Remark : *We have already worked problems to find the Fourier series of* $f(x)$ *directly by applying Euler's formulae when the interval of x is* $(-\pi, \pi)$ *or* $(0, 2\pi)$. *However,* ***when the interval of x is*** $(-\pi, \pi)$ *and also if* $f(x)$ *is either even or odd we can use the results as tabulated above to find the Fourier coefficients thus making the computation work much easier.*

WORKED PROBLEMS

[*As a matter of comparison we shall obtain the Fourier coefficients in three of the already worked problems using the concept of even and odd functions*]

Referring to Problem - 2 : $f(x) = x^2$ in $-\pi \le x \le \pi$

$f(x) = x^2$ and $f(-x) = (-x)^2 = x^2 = f(x)$. Hence $f(x)$ is even. So we have

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx, \quad b_n = 0$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx.$$

Applying Bernoulli's rule,

$$a_n = \frac{2}{\pi}\left[x^2\left(\frac{\sin nx}{n}\right) - (2x)\left(\frac{-\cos nx}{n^2}\right) + 2\left(\frac{-\sin nx}{n^3}\right)\right]_0^{\pi}$$

$$a_n = \frac{4}{\pi n^2}\left[x\cos nx\right]_0^{\pi} = \frac{4}{\pi n^2}(\pi\cos n\pi - 0) = \frac{4(-1)^n}{n^2}$$

Referring to Problem - 10.

$$f(x) = \begin{cases} -k \text{ in } -\pi < x < 0 \\ +k \text{ in } 0 < x < \pi \end{cases}$$

Let $f(x) = \begin{cases} \phi(x) \text{ in } -\pi < x < 0 \\ \psi(x) \text{ in } 0 < x < \pi \end{cases}$ where,

$\phi(x) = -k, \psi(x) = +k$. Now $\phi(-x) = -k = -\psi(x)$

Therefore $f(x)$ is odd.

Hence we have $a_0 = 0, \quad a_n = 0, \quad b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\sin nx\,dx$

Now $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} k\sin nx\,dx = \dfrac{2k}{\pi}\left[\dfrac{-\cos nx}{n}\right]_0^{\pi} = \dfrac{-2k}{\pi n}(\cos n\pi - 1)$

$$b_n = \frac{2k}{\pi n}\left[1 - (-1)^n\right]$$

Referring to Problem - 11. $f(x) = \begin{cases} 1 + \dfrac{2x}{\pi} & \text{in } -\pi < x < 0 \\ 1 - \dfrac{2x}{\pi} & \text{in } 0 < x < \pi \end{cases}$

If $\phi(x) = 1 + \dfrac{2x}{\pi}$ and $\psi(x) = 1 - \dfrac{2x}{\pi}$ in $f(x)$, we have

$$\phi(-x) = 1 - \frac{2x}{\pi} = \psi(x). \text{ Therefore } f(x) \text{ is even.}$$

Hence we have $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^\pi f(x)\,dx$, $a_n = \dfrac{2}{\pi}\displaystyle\int_0^\pi f(x)\cos nx\,dx$, $b_n = 0$

Now $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^\pi \left(1 - \frac{2x}{\pi}\right)dx = \frac{2}{\pi}\left[x - \frac{x^2}{\pi}\right]_0^\pi = 0$, ; $a_0 = 0$

$$a_n = \frac{2}{\pi}\int_0^\pi \left(1 - \frac{2x}{\pi}\right)\cos nx\,dx.$$

Applying Bernoulli's rule,

$$a_n = \frac{2}{\pi}\left[\left(1 - \frac{2x}{\pi}\right)\left(\frac{\sin nx}{n}\right) - \left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^\pi$$

$$= \frac{-4}{\pi^2 n^2}\left[\cos nx\right]_0^\pi$$

$$a_n = \frac{-4}{\pi^2 n^2}(\cos n\pi - 1) = \frac{4}{\pi^2 n^2}\{1 - (-1)^n\}$$

---

12. *Obtain the Fourier series for $f(x) = \sin(mx)$ in the range $(-\pi, \pi)$ where m is neither zero nor an integer.*

\>> The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

We shall check $f(x) = \sin(mx)$ for even or odd nature.

$$f(-x) = \sin(-mx) = -\sin mx = -f(x)$$

$\therefore$ $f(-x) = -f(x)$ and hence $f(x)$ is odd.

Consequently $a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\ \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} \sin mx\ \sin nx\, dx$$

Using $\sin A\ \sin B = \frac{1}{2}\ \{\cos(A-B) - \cos(A+B)\}$ we get

$$b_n = \frac{2}{\pi}\int_0^{\pi} \frac{1}{2}\ \{\cos(m-n)x - \cos(m+n)x\}\ dx$$

$$= \frac{1}{\pi}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{\pi}$$

Note : *It is important to observe that* $\sin k\pi = 0$ *only when k is an integer. Since m is not an integer by data,* $\sin(m-n)\pi$, $\sin(m+n)\pi$ *are not equal to zero and the simplification is carried out as follows.*

$$b_n = \frac{1}{\pi}\left[\frac{1}{m-n}\ \{\sin(m-n)\pi - \sin 0\} - \frac{1}{m+n}\ \{\sin(m+n)\pi - \sin 0\}\right]$$

$$= \frac{1}{\pi}\left\{\frac{1}{m-n}\ \sin(m\pi - n\pi) - \frac{1}{m+n}\ \sin(m\pi + n\pi)\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{m-n}\ (\sin m\pi\ \cos n\pi - \cos m\pi\ \sin n\pi) - \frac{1}{m+n}\ (\sin m\pi\ \cos n\pi + \cos m\pi\ \sin n\pi)\right\}$$

$$= \frac{1}{\pi}\left\{\sin m\pi\ \cos n\pi\left(\frac{1}{m-n} - \frac{1}{m+n}\right)\right\}$$

(*Here* $\sin n\pi = 0$, *since* $n = 1, 2, \ldots$)

$$b_n = \frac{1}{\pi}\left\{\sin m\pi\ \cos n\pi\left(\frac{2n}{m^2-n^2}\right)\right\}$$

$$b_n = \frac{2n(-1)^n\ \sin m\pi}{m^2-n^2}$$

Thus by substituting the values of a_0, a_n, b_n in (1), the Fourier series is given by

$$\sin(mx) = \sum_{n=1}^{\infty} \frac{2n(-1)^n\ \sin m\pi}{m^2-n^2}\ \sin nx$$

13. *Obtain the Fourier series in $(-\pi, \pi)$ for $f(x) = x\cos x$*

>> The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

We shall check $f(x) = x\cos x$ for even or odd nature.

$$f(-x) = -x\cos(-x) = -x\cos x$$

$\therefore \quad f(-x) = -f(x)$ and hence $f(x)$ is odd.

Consequently $a_0 = 0,\ a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x\cos x\,\sin nx\,dx$$

i.e.,

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\sin nx\cos x\,dx \qquad \ldots (2)$$

Using $\sin A\cos B = \frac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$, we have,

$$b_n = \frac{2}{\pi}\int_0^{\pi} x\cdot\frac{1}{2}\left[\sin(nx+x) + \sin(nx-x)\right]dx$$

$$= \frac{1}{\pi}\left\{\int_0^{\pi} x\sin(n+1)x\,dx + \int_0^{\pi} x\sin(n-1)x\,dx\right\}$$

Applying Bernoulli's rule to each of the integrals,

$$b_n = \frac{1}{\pi}\left[x\cdot\left(-\frac{\cos(n+1)x}{n+1}\right) - 1\cdot\left(-\frac{\sin(n+1)x}{(n+1)^2}\right)\right]_0^{\pi}$$

$$+\frac{1}{\pi}\left[x\cdot\left(-\frac{\cos(n-1)x}{n-1}\right) - 1\cdot\left(-\frac{\sin(n-1)x}{(n-1)^2}\right)\right]_0^{\pi},\ (n \neq 1)$$

$$= \frac{1}{\pi}\left\{\frac{-1}{n+1}\left[\pi\cos(n+1)\pi - 0\right] - \frac{1}{n-1}\left[\pi\cos(n-1)\pi - 0\right]\right\}$$

Here $\sin(n+1)\pi = 0 = \sin(n-1)\pi$, since $n = 1, 2, 3, \ldots$

$$b_n = -\left\{\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}$$

$$= -\left\{\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}\right\}$$

$$= (-1)^n\left\{\frac{1}{n+1} + \frac{1}{n-1}\right\} = \frac{(-1)^n\, 2n}{n^2-1}$$

$$\boldsymbol{b_n = \frac{2n(-1)^n}{n^2-1}} \quad (n \neq 1)$$

We shall now find b_n when $n = 1$. That is to find b_1.

Let us consider b_n as given by (2) :

$$b_n = \frac{2}{\pi}\int_0^\pi x \sin nx \cos x \; dx. \quad \text{Putting } n = 1 \text{ we get,}$$

$$b_1 = \frac{2}{\pi}\int_0^\pi x \sin x \cos x \, dx = \frac{2}{\pi}\int_0^\pi x \frac{\sin 2x}{2} dx$$

$$b_1 = \frac{1}{\pi}\int_0^\pi x \sin 2x \, dx.$$

$$b_1 = \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right]_0^\pi$$

$$= \frac{-1}{2\pi}\left| x \cos 2x \right|_0^\pi \qquad \because \sin 2\pi = 0 = \sin 0$$

$$= \frac{-1}{2\pi}(\pi \cos 2\pi - 0) = \frac{-1}{2} \qquad \because \cos 2\pi = 1$$

$$\boldsymbol{b_1 = -1/2}$$

We shall write (1) in the form

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + b_1 \sin x + \sum_{n=2}^{\infty} b_n \sin nx$$

Thus by substituting the values for the Fourier coefficients we have,

$$\boldsymbol{x \cos x = \frac{-1}{2} \sin x + \sum_{n=2}^{\infty} \frac{2n(-1)^n}{n^2-1} \sin nx}$$

---

14. *Expand the function $f(x) = x \sin x$ as a Fourier series in the interval $-\pi \le x \le \pi$. Deduce that*

$$\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots = \frac{\pi - 2}{4}$$

>> The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

We shall check $f(x) = x \sin x$ for even or odd nature.

$$f(-x) = (-x)\sin(-x) = (-x)(-\sin x) = x \sin x$$

$\therefore \quad f(-x) = f(x)$ and hence $f(x)$ is even. Consequently $\boldsymbol{b_n = 0}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx \; ; \; a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x\, dx.$$

$$a_0 = \frac{2}{\pi}\Big[x(-\cos x) - (1)(-\sin x)\Big]_0^{\pi}. \quad \text{But } \sin\pi = 0 = \sin 0$$

i.e., $\quad a_0 = \dfrac{-2}{\pi}(\pi\cos\pi - 0) = 2$

$$a_0/2 = 1$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx\, dx \qquad \ldots (2)$$

Using $\sin A \cos B = \dfrac{1}{2}\left[\sin(A+B) + \sin(A-B)\right]$ we have,

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cdot \frac{1}{2}\left[\sin(x+nx) + \sin(x-nx)\right] dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x \cdot \left[\sin(1+n)x + \sin(1-n)x\right] dx.$$

But $\sin(1-n)x = \sin\left[-(n-1)x\right] = -\sin(n-1)x$

$$\therefore \quad a_n = \frac{1}{\pi}\int_0^{\pi} x \sin(n+1)x\,dx - \frac{1}{\pi}\int_0^{\pi} x\sin(n-1)x\,dx$$

Applying Bernoulli's rule to each of the integrals,

$$a_n = \frac{1}{\pi}\left[x \cdot \frac{-\cos(n+1)x}{n+1} - (1)\cdot\frac{-\sin(n+1)x}{(n+1)^2}\right]_0^{\pi}$$

$$-\frac{1}{\pi}\left[x \cdot \frac{-\cos(n-1)x}{n-1} - (1)\cdot\frac{-\sin(n-1)x}{(n-1)^2}\right]_0^{\pi}, (n \neq 1)$$

But $\sin(n+1)\pi = 0 = \sin(n-1)\pi;\ \sin 0 = 0$

$$a_n = \frac{1}{\pi}\left[\frac{-1}{n+1}\,x\cos(n+1)x\right]_0^{\pi} + \frac{1}{\pi}\left[\frac{x\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{-1}{n+1}\,[\pi\cos(n+1)\pi - 0]\right\} = \frac{1}{\pi}\left\{\frac{\pi}{n-1}\cos(n-1)\pi - 0\right\}$$

$$= -\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}$$

$$= -1\cdot\frac{(-1)^{n+1}}{n+1} + \frac{(-1)^{n-1}}{n-1}, \quad \text{since } \cos k\pi = (-1)^k$$

$$= \frac{(-1)^{n+2}}{n+1} + \frac{(-1)^{n-1}}{n-1} = (-1)^n\left\{\frac{(-1)^2}{n+1} + \frac{(-1)^{-1}}{n-1}\right\}.$$

But $\quad (-1)^{-1} = 1/-1 = -1$

$$a_n = (-1)^n\left\{\frac{1}{n+1} - \frac{1}{n-1}\right\} = (-1)^n\cdot\frac{-2}{n^2-1} = \frac{2(-1)^{n+1}}{n^2-1}$$

$$\mathbf{a_n = \frac{2(-1)^{n+1}}{n^2-1}} \quad \textbf{where } \mathbf{n \neq 1}$$

We shall now find a_n when $n = 1$. That is to find a_1

Let us consider a_n as given by (2). Putting $n = 1$ we have

$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x\cos x\,dx = \frac{2}{\pi}\int_0^{\pi} x\,\frac{\sin 2x}{2}\,dx$$

$$a_1 = \frac{1}{\pi}\int_0^{\pi} x \sin 2x \, dx = -\frac{1}{2} \quad \text{(As in } b_1 \text{ of Problem - 13)}$$

$$\mathbf{a_1 = -1/2}$$

Substituting $b_n = 0$ in (1) we have $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

i.e., $$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

Now substituting the values of $a_0/2, a_1, a_n \ (n \neq 1)$ the required Fourier series is given by

$$\mathbf{x \sin x = 1 - \frac{1}{2}\cos x + \sum_{n=2}^{\infty} \frac{2(-1)^{n+1}}{n^2 - 1} \cos nx}$$

To deduce the series, let us put $x = \pi/2$.

$$\therefore \quad \frac{\pi}{2}\sin\frac{\pi}{2} = 1 - \frac{1}{2}\cos\frac{\pi}{2} + 2\sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}$$

i.e., $$\frac{\pi}{2} = 1 + 2\sum_{n=2}^{\infty}\frac{(-1)^{n+1}}{n^2-1}\cos\frac{n\pi}{2}, \quad \text{since } \sin\frac{\pi}{2} = 1, \cos\frac{\pi}{2} = 0$$

i.e., $$\frac{\pi}{2} - 1 = 2\left(\frac{-1}{3}\cos\pi + \frac{1}{8}\cos\frac{3\pi}{2} - \frac{1}{15}\cos 2\pi + \cdots\right)$$

i.e., $$\frac{\pi - 2}{2} = 2\left(\frac{1}{3} - \frac{1}{15} + - \cdots\right), \text{ since } \cos\pi = -1, \cos 2\pi = 1, \cos(3\pi/2) = 0$$

Thus $$\mathbf{\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots}$$

---

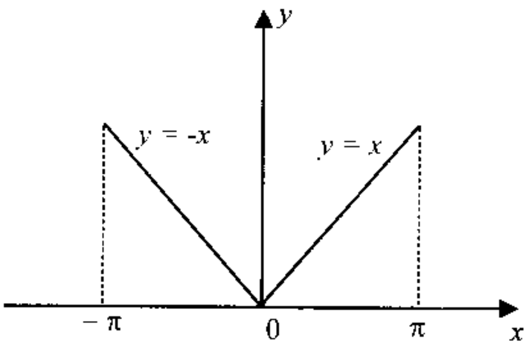
15. *Sketch the graph of the function* $f(x) = |x|$ *in* $-\pi \leq x \leq \pi$ *and obtain its Fourier series.*

Hence deduce that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

>> $f(x) = |x|$ in $-\pi \leq x \leq \pi$ means that the function must be positive in the given interval which consists of negative values and positive values. Hence the given $f(x)$ may be split into the form,

$$f(x) = \begin{cases} -x & \text{in } -\pi \le x \le 0 \\ +x & \text{in } 0 \le x \le \pi \end{cases}$$

The equations $y = x$ and $y = -x$ represent straight lines through the origin with slopes $1, -1$ (*Lines subtending angles* 45^0, 135^0 *with x- axis*) and the graph is as follows.



The Fourier series of $f(x)$ having period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

We shall check $f(x) = |x|$ for even or odd nature.

$$f(-x) = |-x| = |x| = f(x)$$

$\therefore$ $f(-x) = f(x)$ and hence $f(x)$ is even. Consequently $\boldsymbol{b_n = 0}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx\ ;\ a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\, dx$$

Here $f(x) = |x| = x$ for $x \in (0, \pi)$

$$\therefore \quad a_0 = \frac{2}{\pi}\int_0^{\pi} x\, dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{1}{\pi}(\pi^2 - 0) = \pi.$$

$$\boldsymbol{a_0 / 2 = \pi / 2}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx\, dx.$$

Applying Bernoulli's rule,

$$a_n = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - (1)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi n^2}[\cos nx]_0^{\pi}, \quad \text{since } \sin n\pi = 0 = \sin 0$$

$$= \frac{2}{\pi n^2}(\cos n\pi - \cos 0) = \frac{2}{\pi n^2}(\cos n\pi - 1) = \frac{-2}{\pi n^2}(1 - \cos n\pi)$$

$$\mathbf{a_n = \frac{-2}{\pi n^2}\{1-(-1)^n\}}$$

Substituting the values of a_0, a_n, b_n in (1) the Fourier series is given by

$$\mathbf{f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{-2}{\pi n^2}\{1-(-1)^n\}\cos nx}$$

To deduce the series let us put $x = 0$ in the Fourier series.

$$f(0) = \frac{\pi}{2} - \frac{2}{\pi}\sum_1^{\infty}\frac{1}{n^2}\{1-(-1)^n\}\cdot 1$$

i.e., $$0 = \frac{\pi}{2} - \frac{2}{\pi}\sum_1^{\infty}\frac{1}{n^2}\{1-(-1)^n\}$$

$$-\frac{\pi}{2} = -\frac{2}{\pi}\sum_1^{\infty}\frac{1}{n^2}\{1-(-1)^n\} \quad \text{or} \quad \frac{\pi^2}{4} = \sum_1^{\infty}\frac{1}{n^2}\{1-(-1)^n\}.$$

But $$1-(-1)^n = \begin{cases} 1-(+1) = 0 \text{ if } n \text{ is even} \\ 1-(-1) = 2 \text{ if } n \text{ is odd}\end{cases}$$

Hence we get, $$\frac{\pi^2}{4} = \sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2} \quad (2)$$

Thus $$\mathbf{\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots}$$

---

16. *Find the Fourier series in* $(-\pi, \pi)$ *for the function* $f(x)$ *defined by*

$$f(x) = \begin{cases} \pi + x & in\ (-\pi, -\pi/2) \\ \pi/2 & in\ (-\pi/2, \pi/2) \\ \pi - x & in\ (\pi/2, \pi) \end{cases}$$

\>> $f(x)$ is defined in $(-\pi, \pi)$ and the Fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (1)$$

We shall check $f(x)$ for even or odd nature by writing $f(x)$ as follows.

Interval of x	$(-\pi, -\pi/2)$	$(-\pi/2, 0)$	$(0, \pi/2)$	$(\pi/2, \pi)$
$f(x)$	$\pi + x$	$\pi/2$	$\pi/2$	$\pi - x$

i.e., $f(x) = \begin{cases} \phi(x) \text{ in } (-\pi, 0) \\ \psi(x) \text{ in } (0, \pi) \end{cases}$

where $\phi(x) = \pi + x$ or $\pi/2$ and $\psi(x) = \pi - x$ or $\pi/2$

$\therefore$ $\phi(-x) = \pi - x$ or $\pi/2 = \psi(x)$ and hence we conclude that $f(x)$ is even. Consequently $\boldsymbol{b_n = 0}$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\, dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x) \cos nx\, dx$$

$$a_0 = \frac{2}{\pi}\left\{\int_0^{\pi/2} f(x)\, dx + \int_{\pi/2}^{\pi} f(x)\, dx\right\} = \frac{2}{\pi}\left\{\int_0^{\pi/2} \frac{\pi}{2}\, dx + \int_{\pi/2}^{\pi} (\pi - x)\, dx\right\}$$

$$a_0 = \frac{2}{\pi}\left\{\frac{\pi}{2}[x]_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2}\left(\frac{\pi}{2} - 0\right) + \left(\pi^2 - \frac{\pi^2}{2}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^2}{8}\right)\right\} = \frac{2}{\pi} \cdot \frac{3\pi^2}{8} = \frac{3\pi}{4}$$

$\boldsymbol{a_0/2 = 3\pi/8}$

$$a_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} f(x) \cos nx\, dx + \int_{\pi/2}^{\pi} f(x) \cos nx\, dx\right\}$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \frac{\pi}{2} \cos nx\, dx + \int_{\pi/2}^{\pi} (\pi - x) \cos nx\, dx\right\}$$

$$a_n = \frac{2}{\pi}\left\{\frac{\pi}{2}\left[\frac{\sin nx}{n}\right]_0^{\pi/2} + \left[(\pi - x)\frac{\sin nx}{n}\right]_{\pi/2}^{\pi} - \left[(-1)\cdot -\frac{\cos nx}{n^2}\right]_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\frac{\pi}{2n}\left(\sin\frac{n\pi}{2} - 0\right) + \frac{1}{n}\left(0 - \frac{\pi}{2}\sin\frac{n\pi}{2}\right) - \frac{1}{n^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right\}$$

$$= \frac{-2}{\pi n^2}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)$$

$$a_n = \frac{-2}{\pi n^2}\left\{(-1)^n - \cos\frac{n\pi}{2}\right\}$$

Thus by substituting the values of a_0, a_n, b_n in (1), the required Fourier series is given by

$$f(x) = \frac{3\pi}{8} + \sum_{n=1}^{\infty}\frac{-2}{\pi n^2}\left\{(-1)^n - \cos\frac{n\pi}{2}\right\}\cos nx$$

Even and odd nature of $f(x)$ defined in $(0, 2\pi)$

$f(x)$ is said to be even if $f(2\pi - x) = f(x)$ and odd if $f(2\pi - x) = -f(x)$.

We note that $\cos x$ is an even function since $\cos(2\pi - x) = \cos x$ and $\sin x$ is an odd function since $\sin(2\pi - x) = -\sin x$.

Further we have the standard integral property :

$$\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^{a} f(x)\,dx & \text{if } f(x) \text{ is even} \\ 0 & \text{if } f(x) \text{ is odd} \end{cases}$$

If $f(x)$ is a periodic function of period 2π defined in $(0, 2\pi)$ then the Fourier coefficients are given by

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx, \quad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \quad b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

Let us examine the following two cases.

Case - i : $f(x)$ is an even function.

$f(x)\cos nx$ being the product of two even functions is also even and $f(x)\sin nx$ being the product of an even and an odd function is odd.

The Fourier coefficients with the application of the integral property becomes

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx\,,\quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx,\quad b_n = 0$$

Case - ii : $f(x)$ is an odd function .

$f(x)\cos nx$ will be an odd function and $f(x)\sin nx$ will be an even function. Accordingly the Fourier coefficients with the application of the integral property becomes

$$a_0 = 0,\quad a_n = 0,\quad b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

Thus we can conclude that when $x \in (0, 2\pi)$, if $f(x)$ is even $b_n = 0$ and if $f(x)$ is odd $a_0 = 0,\ a_n = 0$

Further it should be observed that the results in respect of the Fourier coefficients involving $f(x)$ defined in $(0, 2\pi)$ are same as in the case of $f(x)$ defined in $(-\pi, \pi)$ where $f(x)$ is even or odd in the relevant interval.

Also it may be noted that if $f(x) = \begin{cases}\phi(x) & \text{in } 0 < x < \pi\\ \psi(x) & \text{in } \pi < x < 2\pi\end{cases}$

we say that $f(x)$ is even if $\phi(2\pi - x) = \psi(x)$ and $f(x)$ is odd if $\phi(2\pi - x) = -\psi(x)$.

WORKED PROBLEMS

[*As a matter of comparision we briefly discuss three of the already worked problems using the concept of even and odd functions*]

Referring to Problem - 1 $f(x) = \dfrac{\pi - x}{2}$ in $(0, 2\pi)$

$$f(2\pi - x) = \frac{\pi - (2\pi - x)}{2} = \frac{-\pi + x}{2} = \frac{-(\pi - x)}{2} = -f(x)$$

$\therefore \quad f(x)$ is odd in $(0, 2\pi)$ and hence $a_0 = 0,\quad a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi}\frac{\pi - x}{2}\sin nx\,dx$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}(\pi - x)\sin nx\,dx = \frac{1}{n}\text{, on integration.}$$

Referring to Problem - 3 : $f(x) = x(2\pi - x)$ in $0 \le x \le 2\pi$

$$f(2\pi - x) = (2\pi - x)(2\pi - \overline{2\pi - x}) = (2\pi - x)x = f(x)$$

$\therefore$ $f(x)$ is even in $(0, 2\pi)$ and hence $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi}(2\pi x - x^2)\,dx = \frac{4\pi^2}{3}, \text{ on integration.}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}(2\pi x - x^2)\cos nx\,dx = \frac{-4}{n^2}, \text{ on integration.}$$

Referring to Problem - 6 $f(x) = \begin{cases} x, & 0 \le x \le \pi \\ 2\pi - x, & \pi \le x \le 2\pi \end{cases}$

Let $f(x) = \begin{cases} \phi(x) = x & \text{in } 0 \le x \le 2\pi \\ \psi(x) = 2\pi - x & \text{in } \pi \le x \le 2\pi \end{cases}$

$$\phi(2\pi - x) = 2\pi - x = \psi(x)$$

$\therefore$ $f(x)$ is even in $(0, 2\pi)$ and hence $b_n = 0$.

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi, \text{ on integration.}$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{-2}{\pi n^2}\{1 - (-1)^n\}, \text{ on integration.}$$

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17. *Obtain the Fourier series representation in* $(0, 2\pi)$ *of the function* $f(x)$ *defined by*

$$f(x) = \begin{cases} x^2 & in \ (0, \pi) \\ -(2\pi - x)^2 & in \ (\pi, 2\pi) \end{cases}$$

>> In the given $f(x)$ let $\phi(x) = x^2$ and $\psi(x) = -(2\pi - x)^2$

Now $\phi(2\pi - x) = (2\pi - x)^2 = -\psi(x)$

$\therefore$ $f(x)$ is odd in $(0, 2\pi)$ and hence $a_0 = 0$, $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\ dx = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx\ dx$$

$$b_n = \frac{2}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - (2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{1}{n}(\pi^2\cos n\pi - 0) + 0 + \frac{2}{n^3}(\cos n\pi - 1)\right]$$

$$b_n = \frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}\left\{(-1)^n - 1\right\}$$

Thus the Fourier series representation of $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty}\left[\frac{2\pi}{n}(-1)^{n+1} + \frac{4}{\pi n^3}\left\{(-1)^n - 1\right\}\right]\sin nx.$$

---

18. *An alternating current after passing through a full wave rectifier has the form $I = I_0\,|\sin t\,|$, $0 < t < 2\pi$ where I_0 is the maximum current. Express I as a Fourier series and hence deduce that* $\frac{1}{2} = \frac{1}{1\cdot 3} + \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} + \cdots$

>> By data $I = f(t) = I_0\,|\sin t\,|$ in $(0, 2\pi)$

i.e., $$I = f(t) = \begin{cases} I_0 \sin t & \text{in } (0, \pi) \\ -I_0 \sin t & \text{in } (\pi, 2\pi) \end{cases}$$

since $\sin t$ is positive if $0 < t < \pi$ and negative if $\pi < t < 2\pi$

Let $$I = f(t) = \begin{cases} \phi(t) = I_0 \sin t & \text{in } (0, \pi) \\ \psi(t) = -I_0 \sin t & \text{in } (\pi, 2\pi) \end{cases}$$

$$\phi(2\pi - t) = I_0 \sin(2\pi - t) = -I_0 \sin t = \psi(t)$$

$\therefore$ $f(t)$ is even in $(0, 2\pi)$ and $b_n = 0$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(t)\,dt, \quad a_n = \frac{2}{\pi}\int_0^{\pi} f(t)\cos nt\,dt$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} I_0 \sin t\,dt = \frac{2I_0}{\pi}\left[-\cos t\right]_0^{\pi} = \frac{-2I_0}{\pi}(-1-1) = \frac{4I_0}{\pi}$$

$$a_0/2 = 2I_0/\pi$$

$$a_n = \frac{2I_0}{\pi}\int_0^{\pi} \sin t \cos nt \, dt$$

Refering to Problem - 8 for the integration process we have

$$a_n = \frac{-2I_0}{\pi(n^2-1)}\left[1+(-1)^n\right] \text{ for } n \neq 1 \text{ and } a_1 = 0$$

We have Fourier series in the form

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nt + \sum_{n=1}^{\infty} b_n \sin nt$$

i.e., $$f(t) = \frac{2I_0}{\pi} + \sum_{n=2}^{\infty} \frac{-2I_0}{\pi(n^2-1)}\left[1+(-1)^n\right]\cos nt$$

Thus $$f(t) = \frac{2I_0}{\pi} - \frac{4I_0}{\pi}\sum_{n=2,4,6\cdots}^{\infty} \frac{\cos nt}{n^2-1}$$

To deduce the series let us put $t = 0$.

$$f(0) = \frac{f(0)+f(2\pi)}{2} = \frac{0+0}{2} = 0$$

The Fourier series becomes,

$$0 = \frac{2I_0}{\pi} - \frac{4I_0}{\pi}\sum_{n=2,4,6\cdots}^{\infty} \frac{1}{n^2-1}$$

i.e., $$\frac{-2I_0}{\pi} = \frac{-4I_0}{\pi}\left[\frac{1}{3}+\frac{1}{15}+\frac{1}{35}+\cdots\right]$$

Thus $$\frac{1}{2} = \frac{1}{1\cdot 3}+\frac{1}{3\cdot 5}+\frac{1}{5\cdot 7}+\cdots\cdots$$

---

19. *Obtain the Fourier series expansion of the function*

$$f(x) = \begin{cases} x & in\ 0 < x < \pi \\ x-2\pi & in\ \pi < x < 2\pi \end{cases}$$

Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$.

>> The Fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

In the given $f(x)$, let $\phi(x) = x$ and $\psi(x) = x - 2\pi$

$$\phi(2\pi - x) = 2\pi - x = -(x - 2\pi) = -\psi(x)$$

$\therefore$ $f(x)$ is odd in $(0, 2\pi)$ and hence $\mathbf{a_0 = 0,\ a_n = 0}$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx\, dx = \frac{2}{\pi}\int_0^{\pi} x \sin nx\, dx$$

$$b_n = \frac{2}{\pi}\left[x\left(\frac{-\cos nx}{n}\right) - (1)\left(\frac{-\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$b_n = \frac{-2}{\pi n}\pi \cos n\pi = \frac{-2}{n}(-1)^n$$

$$\mathbf{b_n = \frac{2(-1)^{n+1}}{n}}$$

Thus the required Fourier series is given by

$$\mathbf{f(x) = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n} \sin nx}$$

To deduce the series let us put $x = \pi/2$. Then $f(x) = \pi/2$ since $f(x) = x$ in $(0, \pi)$. Hence the Fourier series becomes

$$\frac{\pi}{2} = 2\sum_{1}^{\infty} \frac{(-1)^{n+1}}{n} \sin\frac{n\pi}{2}$$

Expanding and noting that $\sin(3\pi/2) = -1$, $\sin(5\pi/2) = 1$, we get

$$\mathbf{\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots\cdots}$$

20. *If* $f(x) = \left(\dfrac{\pi - x}{2}\right)^2$ *in* $0 < x < 2\pi$, *show that*

$$f(x) = \frac{\pi^2}{12} + \sum_{1}^{\infty} \frac{\cos nx}{n^2}.$$ *Hence deduce that*

(i) $$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$$

(ii) $$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$$

(iii) $$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots$$

>> The Fourier series of period 2π is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Consider $f(x) = \left(\dfrac{\pi - x}{2}\right)^2$

$$f(2\pi - x) = \left(\frac{\pi - \overline{2\pi - x}}{2}\right)^2 = \left(\frac{x - \pi}{2}\right)^2 = \left(\frac{\pi - x}{2}\right)^2 = f(x)$$

$\therefore$ $f(x)$ is even in $(0, 2\pi)$ and hence $\boldsymbol{b_n = 0}$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx, \qquad a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} \frac{(\pi - x)^2}{4}\,dx = \frac{1}{2\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{\pi} = \frac{-1}{6\pi}(0 - \pi^3) = \frac{\pi^2}{6}$$

$\boldsymbol{a_0/2 = \pi^2/12}$

$$a_n = \frac{2}{\pi}\int_0^{\pi} \frac{(\pi - x)^2}{4}\cos nx\,dx = \frac{1}{2\pi}\int_0^{\pi} (\pi - x)^2 \cos nx\,dx$$

Applying Bernoulli's rule we get,

$$a_n = \frac{1}{2\pi}\left[(\pi - x)^2\left(\frac{\sin nx}{n}\right) - (-2)(\pi - x)\left(\frac{-\cos nx}{n^2}\right) + (2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{-\pi}$$

$$= \frac{-1}{\pi n^2}\left[(\pi - x)\cos nx\right]_0^{\pi} = \frac{-1}{\pi n^2}(0 - \pi) = \frac{1}{n^2}$$

$$a_n = 1/n^2$$

Thus the required Fourier series is given by

$$f(x) = \left(\frac{\pi - x}{2}\right)^2 = \frac{\pi^2}{12} + \sum_1^{\infty} \frac{\cos nx}{n^2}$$

To deduce the series, we first put $x = 0$

$$f(0) = \frac{f(0) + f(2\pi)}{2} = \frac{(\pi/2)^2 + (-\pi/2)^2}{2} = \frac{\pi^2}{4}$$

Hence the Fourier series becomes,

$$\frac{\pi^2}{4} = \frac{\pi^2}{12} + \sum_1^{\infty} \frac{1}{n^2} \quad \text{or} \quad \frac{\pi^2}{4} - \frac{\pi^2}{12} = \sum_1^{\infty} \frac{1}{n^2}$$

Thus $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ . . . (i)

Next let us put $x = \pi$ in the Fourier series.

$$\therefore \quad 0 = \frac{\pi^2}{12} + \sum_1^{\infty} \frac{\cos n\pi}{n^2} \quad \text{or} \quad \frac{-\pi^2}{12} = \sum_1^{\infty} \frac{(-1)^n}{n^2}$$

Thus $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ . . . (ii)

Adding (i) and (ii) we obtain

$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$ . . . (iii)

1.7 Fourier series of arbitrary period

A function $f(x)$ need not always be defined in an interval of length 2π only. When the length of the interval is other than 2π, we shall denote it by $2l$. A general interval of length $2l$ be $(c,\ c+2l)$. It is important to note that the sine and cosine functions of the form $\sin\left(\frac{\pi x}{l}\right)$ and $\cos\left(\frac{\pi x}{l}\right)$ are periodic functions of period $2l$. It is justified as follows.

Let $\quad F(x) = \sin\left(\frac{\pi x}{l}\right);\quad G(x) = \cos\left(\frac{\pi x}{l}\right)$

Then $\quad F(x+2l) = \sin\left[\frac{\pi}{l}(x+2l)\right] = \sin\left(\frac{\pi x}{l} + 2\pi\right)$

$$= \sin\left(\frac{\pi x}{l}\right) = F(x)$$

Similarly $G(x+2l) = G(x)$.

As already discussed in the article 1.4 the trigonometric series is of the form

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

If $f(x)$ defined in $(c,\ c+2l)$ satisfies Dirichlet's conditions then

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

is called as the ***Fourier series of arbitrary period*** $2l$.

Proceeding on similar lines as in the article 1.4 we can establish Euler's formulae for the Fourier coefficients in the form

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx,\quad a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx,$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Working procedure for problems

- If the period of the given function is other than 2π we first equate the period to $2l$ and obtain the value l.
- We then write the appropriate Fourier series and compute $a_0,\ a_n,\ b_n$ associated with it.

➲ However if $f(x)$ is defined in an interval of the form $(-l, l)$ or $(0, 2l)$ we can compute a_0, a_n, b_n using the concept of even and odd functions taking the following table into consideration.

Nature & condition of $f(x)$	a_0	a_n	b_n
Even function $f(-x) = f(x)$ or $f(2l-x) = f(x)$	$\frac{2}{l}\int_0^l f(x)\,dx$	$\frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$	0
Odd function $f(-x) = -f(x)$ or $f(2l-x) = -f(x)$	0	0	$\frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

WORKED PROBLEMS

21. *Obtain the Fourier series of $f(x) = |x|$ in $(-l, l)$.*

Hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

>> The period of $f(x) = l-(-l) = 2l$ and the Fourier series of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \qquad \ldots (1)$$

We shall check $f(x) = |x|$ for even or odd nature.

$$f(-x) = |-x| = |x| = f(x)$$

Hence $f(x)$ is even and consequently $b_n = 0$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx\,; \quad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_0 = \frac{2}{l}\int_0^l x\,dx = \frac{2}{l}\left[\frac{x^2}{2}\right]_0^l = \frac{1}{l}(l^2-0) = l$$

$$\mathbf{a_0/2 = l/2}$$

$a_n = \frac{2}{l}\int_0^l x\cos\frac{n\pi x}{l}\,dx$. Applying Bernoulli's rule,

$$a_n = \frac{2}{l}\left[x \cdot \frac{\sin \frac{n\pi x}{l}}{(n\pi/l)} - (1) \cdot - \frac{\cos \frac{n\pi x}{l}}{(n\pi/l)^2} \right]_0^l$$

$$= \frac{2}{l} \frac{l^2}{n^2\pi^2}\left[\cos \frac{n\pi x}{l} \right]_0^l = \frac{2l}{n^2\pi^2}(\cos n\pi - 1), \text{ since } \sin n\pi = 0 = \sin 0$$

$$a_n = \frac{-2l}{n^2\pi^2}\left\{ 1 - (-1)^n \right\}$$

Thus the required Fourier series is given by

$$f(x) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{-2l}{n^2\pi^2}\left\{ 1 - (-1)^n \right\} \cos \frac{n\pi x}{l}$$

To deduce the series we shall put $x = 0$ which gives $f(x) = 0$ and the Fourier series becomes,

$$0 = \frac{l}{2} - \frac{2l}{\pi^2} \sum_1^{\infty} \frac{1}{n^2}\left\{ 1 - (-1)^n \right\} \cdot 1$$

i.e.,

$$\frac{-l}{2} = \frac{-2l}{\pi^2} \sum_1^{\infty} \frac{1}{n^2}\left\{ 1 - (-1)^n \right\} \quad \text{or} \quad \frac{\pi^2}{4} = \sum_{n=1}^{\infty} \frac{1}{n^2}\left\{ 1 - (-1)^n \right\}$$

But

$$1 - (-1)^n = \begin{cases} 1 - (+1) = 0 & \text{when } n \text{ is even} \\ 1 - (-1) = 2 & \text{when } n \text{ is odd} \end{cases}$$

$$\therefore \quad \frac{\pi^2}{4} = \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cdot 2 \quad \text{or} \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$$

---

22. *Obtain the Fourier series to represent*

$$f(x) = x - x^2 \quad in \quad -1 < x < 1$$

>> Here period of $f(x) = 1 - (-1) = 2 \quad \therefore 2l = 2$ or $l = 1$

The Fourier series of $f(x)$ having period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos(n\pi x) + \sum_1^{\infty} b_n \sin(n\pi x)$$

Since the interval is $(-1, 1)$ let us check the given function for even or odd nature.

$f(x) = x - x^2 \therefore f(-x) = -x - (-x)^2 = -x - x^2$ which is neither equal to $f(x)$ nor equal to $-f(x)$. So $f(x)$ is neither even nor odd.

We shall find the Fourier coefficients by Euler's formulae.

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}(x-x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1}$$

i.e., $$a_0 = \left(\frac{1}{2} - \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{1}{3}\right) = \frac{-2}{3}$$

$$\boldsymbol{a_0/2 = -1/3}$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(x)\cos(n\pi x)\,dx$$

$$a_n = \int_{-1}^{1}(x-x^2)\cos(n\pi x)\,dx.$$ Applying Bernoulli's rule,

$$a_n = \left[(x-x^2)\frac{\sin(n\pi x)}{n\pi} - (1-2x)\cdot -\frac{\cos(n\pi x)}{n^2\pi^2} + (-2)\cdot -\frac{\sin(n\pi x)}{n^3\pi^3}\right]_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}\left[(1-2x)\cos(n\pi x)\right]_{-1}^{1}, \quad \text{since } \sin n\pi = 0$$

$$= \frac{1}{n^2\pi^2}\left[-\cos n\pi - 3\cos n\pi\right] = \frac{-4\cos n\pi}{n^2\pi^2} = \frac{-4(-1)^n}{n^2\pi^2}$$

$$\boldsymbol{a_n = \frac{4(-1)^{n+1}}{n^2\pi^2}}$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(n\pi x)\,dx = \int_{-1}^{1}(x-x^2)\sin(n\pi x)\,dx$$

$$= \left[(x-x^2)\cdot\frac{-\cos(n\pi x)}{n\pi} - (1-2x)\cdot\frac{-\sin(n\pi x)}{n^2\pi^2} + (-2)\frac{\cos(n\pi x)}{n^3\pi^3}\right]_{-1}^{1}$$

$$b_n = \frac{-1}{n\pi}\left[0-(-2\cos n\pi)\right] - \frac{2}{n^3\pi^3}(\cos n\pi - \cos n\pi) = \frac{-2}{n\pi}(-1)^n$$

$$\boldsymbol{b_n = \frac{2}{n\pi}(-1)^{n+1}}$$

Thus the required Fourier series is given by

$$\boldsymbol{f(x) = \frac{-1}{3} + \frac{4}{\pi^2}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n^2}\cos n\pi x + \frac{2}{\pi}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{n}\sin n\pi x}$$

---

23. *Draw the graph of the function*

$$f(x) = \begin{cases} \pi x \ in \ 0 \le x \le 1 \\ \pi(2-x) \ in \ 1 \le x \le 2 \end{cases}$$

and also show that the Fourier expansion of the function $f(x)$ is

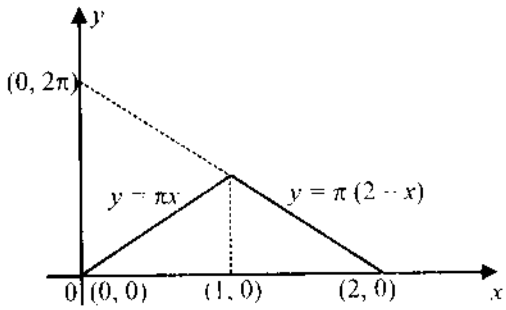
$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$$

>> Graph of $f(x) = y$

$f(x) = \pi x$ or $y = \pi x$ is a line passing through the origin in [0, 1]

$$f(x) = \pi(2-x) \text{ or } y = \pi(2-x) \text{ or } \pi x + y = 2\pi \text{ or } \frac{x}{2} + \frac{y}{2\pi} = 1$$

in [1, 2], which is a straight line passing through $(2, 0)$ and $(0, 2\pi)$. The graph is as follows.



Here $f(x)$ is defined in [0, 2] and period of $f(x) = 2 - 0 = 2$

$\therefore$ $2l = 2$ or $l = 1$.

The Fourier series of $f(x)$ having period 2 is given by

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos(n\pi x) + \sum_1^{\infty} b_n \sin(n\pi x)$$

$$a_0 = \frac{1}{1}\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$a_0 = \int_0^1 \pi x\,dx + \int_1^2 \pi(2-x)\,dx$$

$$a_0 = \pi \left\{ \left[\frac{x^2}{2} \right]_0^1 + \left[2x - \frac{x^2}{2} \right]_1^2 \right\}$$

$$= \pi \left\{ \left(\frac{1}{2} - 0 \right) + (4-2) - \left(2 - \frac{1}{2} \right) \right\} = \pi$$

$$a_0/2 = \pi/2$$

$$a_n = \frac{1}{1} \int_0^2 f(x) \cos(n\pi x)\, dx$$

$$= \int_0^1 f(x) \cos(n\pi x)\, dx + \int_1^2 f(x) \cos(n\pi x)\, dx$$

$$= \int_0^1 \pi x \cos(n\pi x)\, dx + \int_1^2 \pi(2-x) \cos(n\pi x)\, dx$$

$$= \pi \left\{ \int_0^1 x \cos(n\pi x)\, dx + \int_1^2 (2-x) \cos(n\pi x)\, dx \right\}$$

Applying Bernoulli's rule to both the integrals

$$a_n = \pi \left\{ \left[x \cdot \frac{\sin(n\pi x)}{n\pi} - 1 \cdot -\frac{\cos(n\pi x)}{n^2 \pi^2} \right]_0^1 + \right.$$

$$\left. \left[(2-x) \frac{\sin(n\pi x)}{n\pi} - (-1) \cdot -\frac{\cos(n\pi x)}{n^2 \pi^2} \right]_1^2 \right\}$$

$$= \frac{\pi}{n^2 \pi^2} \left\{ \left[\cos n\pi x \right]_0^1 - \left[\cos n\pi x \right]_1^2 \right\}, \quad \text{since } \sin n\pi = 0 = \sin 0$$

$$= \frac{1}{n^2 \pi} (\cos n\pi - \cos 0 - \cos 2n\pi + \cos n\pi). \text{ But } \cos 2n\pi = 1 = \cos 0$$

$$a_n = \frac{1}{n^2 \pi} (-2 + 2 \cos n\pi)$$

$$a_n = \frac{-2}{\pi n^2}\left\{1-(-1)^n\right\}$$

$$b_n = \frac{1}{1}\int_0^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 f(x)\sin(n\pi x)\,dx + \int_1^2 f(x)\sin(n\pi x)\,dx$$

$$= \int_0^1 \pi x\sin(n\pi x)\,dx + \int_1^2 \pi(2-x)\sin(n\pi x)\,dx$$

$$= \pi\left\{\left[x\cdot\frac{-\cos(n\pi x)}{n\pi} - (1)\cdot\frac{-\sin(n\pi x)}{n^2\pi^2}\right]_0^1 + \left[(2-x)\cdot\frac{-\cos(n\pi x)}{n\pi} - (-1)\cdot\frac{-\sin(n\pi x)}{n^2\pi^2}\right]_1^2\right\}$$

$$= \frac{-\pi}{n\pi}\left\{\left[x\cos(n\pi x)\right]_0^1 + \left[(2-x)\cos(n\pi x)\right]_1^2\right\}$$

$$= \frac{-1}{n}\left\{(\cos n\pi - 0) + (0 - \cos n\pi)\right\} = 0$$

$$b_n = 0$$

The required Fourier series is given by

$$f(x) = \frac{\pi}{2} - \frac{2}{\pi}\sum_{n=1}^{\infty}\left\{\frac{1-(-1)^n}{n^2}\right\}\cos(n\pi x)$$

But $1-(-1)^n = \begin{cases}1-(+1) = 0 & \text{if } n \text{ is even}\\ 1-(-1) = 2 & \text{if } n \text{ is odd}\end{cases}$

Hence $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1,3,5,\ldots}^{\infty}\frac{\cos(n\pi x)}{n^2}$

Thus $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos\pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \cdots\right)$

Now putting $x = 0$ we have $f(x) = 0$ since $f(x) = \pi x$ in [0, 1]

The Fourier series becomes

$$0 = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right) \text{ or } -\frac{\pi}{2} = \frac{-4}{\pi}\left(\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right)$$

Thus $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

Aliter : (*Using the concept of even and odd functions*)

$f(x)$ is defined in $(0, 2)$ which is of the form $(0, 2l)$.

In the given $f(x)$ if $\phi(x) = \pi x$ and $\psi(x) = \pi(2-x)$,

$$\phi(2l - x) = \phi(2-x) = \pi(2-x) = \psi(x)$$

$\therefore$ $f(x)$ is even in $(0, 2)$ and hence $b_n = 0$

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx, \quad a_n = \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$a_0 = 2\int_0^1 \pi x\,dx = \pi \text{ , on integration.}$$

$$a_n = 2\int_0^1 \pi x\cos(n\pi x)\,dx$$

$$= 2\pi\int_0^1 x\cos(n\pi x)\,dx = \frac{-2}{\pi n^2}\left[1-(-1)^n\right] \text{ , on integration.}$$

---

24. *Obtain the Fourier series for the function*

$$f(x) = \begin{cases} 1 + \dfrac{4x}{3} & in\ \dfrac{-3}{2} < x \le 0 \\ 1 - \dfrac{4x}{3} & in\ 0 \le x < \dfrac{3}{2} \end{cases}$$

Hence deduce that $\dfrac{\pi^2}{8} = \dfrac{1}{1^2} + \dfrac{1}{3^2} + \dfrac{1}{5^2} + \cdots$

>> $f(x)$ is defined in the interval $(-3/2, 3/2)$

$\therefore$ period of $f(x) = 3/2 - (-3/2) = 3$. $2l = 3$ or $l = 3/2$

We shall check $f(x)$ for even or odd nature.

If $\phi(x) = 1 + \frac{4x}{3}, \quad \phi(-x) = 1 - \frac{4x}{3} = \psi(x)$

$\therefore$ $f(x)$ is an even function. Consequently $b_n = 0$

The Fourier series of $f(x)$ having period 3 is given by

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos\left(\frac{n\pi x}{3/2}\right) + \sum_1^\infty b_n \sin\left(\frac{n\pi x}{3/2}\right)$$

i.e., $$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos\left(\frac{2n\pi x}{3}\right) + \sum_1^\infty b_n \sin\left(\frac{2n\pi x}{3}\right)$$

Since $f(x)$ is even we have,

$$a_0 = \frac{2}{3/2}\int_0^{3/2} f(x)\,dx, \quad \text{since } a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

i.e., $$a_0 = \frac{4}{3}\int_0^{3/2}\left(1 - \frac{4x}{3}\right)dx$$

$$= \frac{4}{3}\left[x - \frac{2x^2}{3}\right]_0^{3/2} = \frac{4}{3}\left\{\left(\frac{3}{2} - \frac{2}{3}\cdot\frac{9}{4}\right) - 0\right\} = 0$$

$$\mathbf{a_0/2 = 0}$$

$$a_n = \frac{2}{3/2}\int_0^{3/2} f(x)\cos\left(\frac{2n\pi x}{3}\right)dx \quad \text{since } a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\,dx$$

$$a_0 = \frac{4}{3}\int_0^{3/2}\left(1 - \frac{4x}{3}\right)\cos\left(\frac{2n\pi x}{3}\right)dx. \text{ Applying Bernoulli's rule}$$

$$a_n = \frac{4}{3}\left[\left(1 - \frac{4x}{3}\right)\frac{\sin\frac{2n\pi x}{3}}{2n\pi/3} - \left(\frac{-4}{3}\right)\cdot - \frac{\cos\frac{2n\pi x}{3}}{(2n\pi/3)^2}\right]_0^{3/2}$$

$$= \frac{4}{3}\cdot\frac{-4}{3}\cdot\frac{9}{4n^2\pi^2}\left[\cos\left(\frac{2n\pi x}{3}\right)\right]_0^{3/2} = \frac{-4}{n^2\pi^2}(\cos n\pi - 1)$$

$$a_n = \frac{4}{n^2\pi^2}\{1 - (-1)^n\} \text{ or } a_n = 8/n^2\pi^2 \text{ where } n = 1, 3, 5, \ldots$$

Thus the required Fourier series is given by

$$f(x) = \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos\left(\frac{2n\pi x}{3}\right)$$

Putting $x = 0$ we get $f(x) = 1$. The Fourier series becomes

$$1 = \frac{8}{\pi^2} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \quad \text{or} \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

---

25. *Obtain the Fourier series for* $f(x) = e^{-x}$ *in the interval* $0 < x < 2$.

>> The period of $f(x) = 2 - 0 = 2 \quad \therefore \quad 2l = 2$ or $l = 1$.

The relevant Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos n\pi x + \sum_1^{\infty} b_n \sin n\pi x \qquad \ldots(1)$$

$$a_0 = \int_0^2 f(x)\,dx, \quad a_n = \int_0^2 f(x) \cos n\pi x\,dx, \quad b_n = \int_0^2 f(x) \sin \pi x\,dx$$

(*In each of the above integrals* $1/l = 1/1 = 1$)

$$a_0 = \int_0^2 e^{-x}\,dx = \left[-e^{-x}\right]_0^2 = -(e^{-2} - 1) = 1 - \frac{1}{e^2} = \frac{e^2 - 1}{e^2}$$

$$a_0/2 = (e^2 - 1)/2e^2$$

$$a_n = \int_0^2 e^{-x} \cos n\pi x\,dx$$

We have $\int e^{ax} \cos bx\,dx = \dfrac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)$

$$\therefore \qquad a_n = \left[\frac{e^{-x}}{1 + n^2\pi^2}(-\cos n\pi x + n\pi \sin n\pi x)\right]_0^2$$

$$= \frac{-1}{1 + n^2\pi^2}\left[e^{-x} \cos n\pi x\right]_0^2 \qquad \because \sin 2n\pi = 0 = \sin 0$$

$$= \frac{-1}{1 + n^2\pi^2}\left\{e^{-2} \cos 2n\pi - 1\right\} = \frac{-1}{1 + n^2\pi^2}\left(\frac{1}{e^2} - 1\right)$$

$$a_n = \frac{e^2 - 1}{e^2(1 + n^2\pi^2)}$$

$$b_n = \int_0^2 e^{-x} \sin n\pi x \, dx$$

We have $\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$

$$\therefore \quad b_n = \left[\frac{e^{-x}}{1 + n^2\pi^2} (- \sin n\pi x - n\pi \cos n\pi x) \right]_0^2$$

$$= \frac{-n\pi}{1 + n^2\pi^2} \left[e^{-x} \cos n\pi x \right]_0^2 = \frac{-n\pi}{1 + n^2\pi^2} \left(\frac{1}{e^2} - 1 \right)$$

$$b_n = \frac{n\pi(e^2 - 1)}{e^2(1 + n^2\pi^2)}$$

Thus by substituting the values of a_0, a_n, b_n in (1), the Fourier series is given by

$$f(x) = \frac{e^2 - 1}{2e^2} + \sum_1^\infty \frac{e^2 - 1}{e^2(1 + n^2\pi^2)} \cos n\pi x + \sum_1^\infty \frac{n\pi(e^2 - 1)}{e^2(1 + n^2\pi^2)} \sin n\pi x$$

---

26. *Find the Fourier series of the periodic function defined by* $f(x) = 2x - x^2$ *in the interval* $0 < x < 3$

>> The period of $f(x) = 3 - 0 = 3 \quad \therefore \ 2l = 3$ or $l = 3/2$

The Fourier series of period $2l$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^\infty b_n \sin \frac{n\pi x}{l}$$

The relevant Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos \frac{2n\pi x}{3} + \sum_1^\infty b_n \sin \frac{2n\pi x}{3} \qquad \ldots (1)$$

We shall find Fourier coefficients from Euler's formulae for the interval (0, 3) with reference to the Fourier series (1). That is

$$a_0 = \frac{2}{3} \int_0^3 f(x)\, dx, \quad a_n = \frac{2}{3} \int_0^3 f(x) \cos \frac{2n\pi x}{3} dx, \quad b_n = \frac{2}{3} \int_0^3 f(x) \sin \frac{2n\pi x}{3} dx$$

In each of the above integrals $1/l = 1/(3/2) = 2/3$

$$a_0 = \frac{2}{3}\int_0^3 (2x - x^2)\ dx$$

$$= \frac{2}{3}\left[x^2 - \frac{x^3}{3}\right]_0^3 = \frac{2}{3}\ \{(9-9) - 0\} = 0$$

$$\mathbf{a_0 / 2 = 0}$$

$$a_n = \frac{2}{3}\int_0^3 (2x - x^2)\ \cos\frac{2n\pi x}{3}\ dx.$$

Applying Bernoulli's rule,

$$a_n = \frac{2}{3}\left[(2x - x^2)\ \frac{\sin\frac{2n\pi x}{3}}{2n\pi/3} - (2-2x)\cdot\frac{-\cos\frac{2n\pi x}{3}}{(2n\pi/3)^2} + (-2)\cdot\frac{-\sin\frac{2n\pi x}{3}}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\ \frac{9}{4n^2\pi^2}\left[(2-2x)\cos\frac{2n\pi x}{3}\right]_0^3$$

The first and third terms vanish since $\sin 2n\pi = 0 = \sin 0$

$$a_n = \frac{3}{2n^2\pi^2}\ \{(2-6)\ \cos 2n\pi - (2-0)\ \cos 0\} = \frac{3}{2n^2\pi^2}\ (-4-2)$$

$$\mathbf{a_n = -9/n^2\pi^2}$$

$$b_n = \frac{2}{3}\int_0^3 (2x - x^2)\ \sin\frac{2n\pi x}{3}\ dx.$$ Applying Bernoulli's rule,

$$b_n = \frac{2}{3}\left[(2x - x^2)\cdot\frac{-\cos\frac{2n\pi x}{3}}{2n\pi/3} - (2-2x)\cdot\frac{-\sin\frac{2n\pi x}{3}}{(2n\pi/3)^2} + (-2)\cdot\frac{\cos\frac{2n\pi x}{3}}{(2n\pi/3)^3}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}\left\{(2x - x^2)\ \cos\frac{2n\pi x}{3}\right\} - \frac{54}{8n^3\pi^3}\left\{\cos\frac{2n\pi x}{3}\right\}\right]_0^3$$

$$= \frac{2}{3}\left[\frac{-3}{2n\pi}\ \{(6-9)\ \cos 2n\pi - 0\} - \frac{54}{8n^3\pi^3}\ \{\cos 2n\pi - \cos 0\}\right]_0^3$$

$$= \frac{2}{3}\left\{\frac{-3}{2n\pi}\ (-3)\right\} = \frac{3}{n\pi}$$

$$b_n = 3/n\pi$$

Thus by substituting the values of a_0, a_n, b_n in (1) the required Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2} \cos\frac{2n\pi x}{3} + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin\frac{2n\pi x}{3}$$

---

27. *Obtain the Fourier series for the function* $f(x) = 2x - x^2$ *in* $0 \le x \le 2$

>> Comparing the given interval $(0, 2)$ with $(0, 2l)$ we have

$2l = 2$ or $l = 1$. By data $f(x) = x(2-x)$

$f(2l-x) = f(2-x) = (2-x)(2-\overline{2-x}) = (2-x)x = f(x)$

$\therefore$ $f(x)$ is even in $(0, 2)$ and hence $b_n = 0$.

$$a_0 = \frac{2}{1}\int_0^1 f(x)\,dx, \quad a_n = \frac{2}{1}\int_0^1 f(x)\cos(n\pi x)\,dx$$

$$a_0 = 2\int_0^1 (2x - x^2)\,dx = 2\left[x^2 - \frac{x^3}{3}\right]_0^1 = 2\left[\left(1-\frac{1}{3}\right)-0\right] = \frac{4}{3}$$

$$a_0/2 = 2/3$$

$a_n = 2\int_0^1 (2x - x^2)\cos(n\pi x)\,dx$. Applying Bernoulli's rule,

$$a_n = 2\left[(2x-x^2)\frac{\sin(n\pi x)}{n\pi} - (2-2x)\cdot\frac{-\cos(n\pi x)}{n^2\pi^2} + (-2)\cdot\frac{-\sin(n\pi x)}{n^3\pi^3}\right]_0^1$$

$$= \frac{2}{n^2\pi^2}\left[(2-2x)\cos(n\pi x)\right]_0^1 = \frac{2}{n^2\pi^2}\left[0 - 2\cos 0\right] = \frac{-4}{n^2\pi^2}$$

$$a_n = -4/n^2\pi^2$$

The Fourier series of period 2 is represented by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos(n\pi x) + \sum_{n=1}^{\infty} b_n\sin(n\pi x)$$

Thus the required Fourier series is given by

9

$$f(x) = 2x - x^2 = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{-4}{n^2\pi^2} \cos(n\pi x)$$

Note : *The problem can be worked directly also.*

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28. *Obtain the Fourier series of the saw-tooth function* $f(t) = Et/T$ *for* $0 < t < T$ *given that* $f(t+T) = f(t)$ *for all* $t > 0$

>> We have $2l = T$ or $l = T/2$ and the associated Fourier series of period $2l = T$ is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{T/2}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{T/2}\right)$$

i.e., $$f(t) = \frac{a_0}{2} + \sum_{1}^{\infty} a_n \cos\left(\frac{2n\pi t}{T}\right) + \sum_{1}^{\infty} b_n \sin\left(\frac{2n\pi t}{T}\right)$$

We compute Fourier coefficients by Euler's formulae.

$$a_0 = \frac{1}{T/2}\int_0^T f(t)\,dt, \quad a_n = \frac{1}{T/2}\int_0^T f(t)\cos\left(\frac{2n\pi t}{T}\right)dt$$

$$b_n = \frac{1}{T/2}\int_0^T f(t)\sin\left(\frac{2n\pi t}{T}\right)dt$$

$$a_0 = \frac{2}{T}\int_0^T \frac{Et}{T}\,dt = \frac{2E}{T^2}\left[\frac{t^2}{2}\right]_0^T = \frac{E}{T^2}(T^2 - 0) = E$$

$$\mathbf{a_0/2 = E/2}$$

$$a_n = \frac{2}{T}\int_0^T \frac{Et}{T}\cos\frac{2n\pi t}{T}\,dt = \frac{2E}{T^2}\int_0^T t\cos\frac{2n\pi t}{T}\,dt$$

$$a_n = \frac{2E}{T^2}\left[t \cdot \frac{\sin\frac{2n\pi t}{T}}{(2n\pi/T)} - 1 \cdot \frac{-\cos\frac{2n\pi t}{T}}{(2n\pi/T)^2}\right]_0^T$$

$$= \frac{2E}{T^2} \cdot \frac{T^2}{4n^2\pi^2}(\cos 2n\pi - \cos 0) = 0$$

$$\mathbf{a_n = 0}$$

$$b_n = \frac{2}{T} \cdot \frac{E}{T} \int_0^T t \sin \frac{2n\pi t}{T} \, dt$$

$$= \frac{2E}{T^2} \left[t \cdot \frac{-\cos \frac{2n\pi t}{T}}{(2n\pi/T)} - 1 \cdot \frac{-\sin \frac{2n\pi t}{T}}{(2n\pi/T)^2} \right]_0^T$$

$$= \frac{-E}{n\pi T} \left[t \cos \frac{2n\pi t}{T} \right]_0^T = \frac{-E}{n\pi T} (T \cos 2n\pi - 0) = \frac{-E}{n\pi}$$

$$\boldsymbol{b_n = -E/n\pi}$$

Thus the required Fourier series is given by

$$\boldsymbol{f(t) = \frac{E}{2} - \frac{E}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{2n\pi t}{T}}$$

---

29. *If* $f(x) = \begin{cases} 2 - x & in \ 0 \le x \le 4 \\ x - 6 & in \ 4 \le x \le 8 \end{cases}$

Express $f(x)$ *as a Fourier series and hence deduce that* $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

>> Comparing the given interval $(0, 8)$ with $(0, 2l)$ we have $2l = 8$ or $l = 4$. The Fourier series having period 8 is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right)$$

In the given $f(x)$ let $\phi(x) = 2 - x$, $\psi(x) = x - 6$

Now $\phi(2l - x) = \phi(8 - x) = 2 - (8 - x) = x - 6 = \psi(x)$

$\therefore$ $f(x)$ is even in $(0, 8)$ and hence $\boldsymbol{b_n = 0}$

$$a_0 = \frac{2}{4} \int_0^4 f(x)\, dx, \quad a_n = \frac{2}{4} \int_0^4 f(x) \cos\left(\frac{n\pi x}{4}\right) dx$$

$$a_0 = \frac{1}{2} \int_0^4 (2 - x)\, dx = \frac{1}{2} \left[2x - \frac{x^2}{2} \right]_0^4 = \frac{1}{2} [(8 - 8) - 0] = 0$$

$a_0 / 2 = 0$

$a_n = \frac{1}{2} \int_0^4 (2-x) \cos \frac{n\pi x}{4} dx$. Applying Bernoulli's rule,

$$a_n = \frac{1}{2} \left[\frac{(2-x) \sin \frac{n\pi x}{4}}{(n\pi/4)} - (-1) \cdot \frac{-\cos \frac{n\pi x}{4}}{(n\pi/4)^2} \right]_0^4$$

$$= \frac{-8}{n^2 \pi^2} \left[\cos \frac{n\pi x}{4} \right]_0^4 = \frac{-8}{n^2 \pi^2} \cos n\pi - 1$$

$$a_n = \frac{8}{n^2 \pi^2} \left| 1 - (-1)^n \right| = \frac{16}{n^2 \pi^2} \text{ where } n = 1, 3, 5, \cdots$$

Thus the required Fourier series is given by

$$f(x) = \sum_{n=1,3,5\cdots}^{\infty} \frac{16}{n^2 \pi^2} \cos \left(\frac{n\pi x}{4} \right)$$

To deduce the series we put $x = 0$. $f(x) = 2 - 0 = 2$ and the Fourier series becomes

$$2 = \frac{16}{\pi^2} \sum_{n=1,3,5\cdots}^{\infty} \frac{1}{n^2} \cdot 1 \quad \text{or} \quad \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Equivalently we have $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

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30. *Find the Fourier expansion of the function $f(x)$ defined by*

$$f(x) = \begin{cases} 0 & in \ -2 < x < -1 \\ 2 & in \ -1 < x < 1 \\ 0 & in \ \ 1 < x < 2 \end{cases}$$

>> $f(x)$ is defined in $(-2, 2)$ and period of $f(x) = 2 - (-2) = 4$

We have $2l = 4$ or $l = 2$. The associated Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{2} + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{2} \qquad \ldots (1)$$

The given $f(x)$ can be written as follows.

Interval of x	$(-2, -1)$	$(-1, 0)$	$(0, 1)$	$(1, 2)$
$f(x)$	0	2	2	0

Let $f(x) = \begin{cases} \phi(x) \text{ in } (-2, 0) \\ \psi(x) \text{ in } (0, 2) \end{cases}$

where $\phi(x) = 0$ or 2, $\psi(x) = 2$ or 0. Obviously $\phi(-x) = \psi(x)$

$\therefore$ $f(x)$ is even and consequently $b_n = 0$

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx, \quad a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$a_0 = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx = \int_0^1 2\,dx + 0 = |\,2x\,|_0^1 = 2$$

$a_0/2 = 1$

$$a_n = \int_0^1 f(x)\cos\frac{n\pi x}{2}\,dx + \int_1^2 f(x)\cos\frac{n\pi x}{2}\,dx$$

$$= \int_0^1 2\cos\frac{n\pi x}{2}\,dx + 0$$

$$a_n = 2\left[\frac{\sin\frac{n\pi x}{2}}{(n\pi/2)}\right]_0^1 = \frac{4}{n\pi}\sin\left(\frac{n\pi}{2}\right)$$

Thus the required Fourier series is given by

$$f(x) = 1 + \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\left(\frac{n\pi}{2}\right)\cos\frac{n\pi x}{2}$$

EXERCISES

Find the Fourier series of the following functions over the indicated intervals

1. $f(x) = -1 + x;\ -\pi < x < \pi$
2. $f(x) = x^2;\ 0 < x < 2\pi$
3. $f(x) = \pi^2 - x^2;\ -\pi \le x \le \pi$. Deduce that

   (a) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - + \cdots = \frac{\pi^2}{12}$ (b) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \frac{\pi^2}{6}$
4. $f(x) = \frac{x}{12}(\pi^2 - x^2);\ -\pi < x < \pi$

5. $f(x) = x \sin x$; $(0, 2\pi)$. Deduce that

(a) $\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \cdots = \frac{\pi - 2}{4}$

(b) $\sum_{n=2}^{\infty} \frac{1}{n^2 - 1} = \frac{3}{4}$

6. $f(x) = \begin{cases} x - \frac{\pi}{2} & \text{in } (-\pi, 0) \\ x + \frac{\pi}{2} & \text{in } (0, \pi) \end{cases}$

7. $f(x) = \begin{cases} \pi - x & \text{in } 0 \le x \le \pi \\ x - \pi & \text{in } \pi \le x \le 2\pi \end{cases}$

8. $f(x) = \begin{cases} -\cos x & \text{in } (-\pi, 0) \\ \cos x & \text{in } (0, \pi) \end{cases}$

9. $f(x) = \begin{cases} -1 & \text{in } -\pi < x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{in } 0 < x < \pi \end{cases}$

Hence deduce that $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$

10. $f(x) = l^2 + x^2$ in $-l \le x \le l$.

Deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

11. $f(x) = 1 - 2|x|$ in $-1 \le x \le 1$

12. $f(x) = x^2 - x$ in $(-2, 2)$

13. $f(x) = x - x^2$ in $(0, 1)$.

Deduce that $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

14. $f(x) = \begin{cases} \frac{l}{2} - x & \text{in } (-l, 0) \\ \frac{l}{2} + x & \text{in } (0, l) \end{cases}$

Deduce that $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

15. $$f(x) = \begin{cases} x^2 & \text{in } (0,\ 1) \\ -(2-x)^2 & \text{in } (1,\ 2) \end{cases}$$

16. $$f(x) = \begin{cases} 8 & \text{in } 0 < x < 2 \\ -8 & \text{in } 2 < x < 4 \end{cases}$$

17. $$f(x) = \begin{cases} 0 & \text{in } -2 < x \le -1 \\ 1+x & \text{in } -1 < x < 0 \\ (1-x) & \text{in } 0 \le x < 1 \\ 0 & \text{in } 1 \le x < 2 \end{cases}$$

18. Prove that if $-\pi \le x \le \pi$ and a is not an integer

$$\cos ax = \frac{2a}{\pi} \sin a\pi \left\{ \frac{1}{2a^2} + \sum_1^{\infty} \frac{(-1)^{n+1}}{n^2 - a^2} \cos nx \right\}$$

Hence show that $\dfrac{1 - a\pi \cot a\pi}{2a^2} = \displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2}$

19. If $0 < x < 2\pi$ show that $x \cos x = \pi \cos x - \dfrac{1}{2} \sin x - 2 \displaystyle\sum_{n=2}^{\infty} \frac{n}{n^2 - 1} \sin nx$

20. Show that the Fourier series representation in $(0,\ 2T)$ of the half wave rectifier $f(t)$ defined by

$$f(t) = \begin{cases} E \sin \dfrac{\pi t}{T} & \text{in } (0, T) \\ 0 & \text{in } (T, 2T) \end{cases} \quad \text{is}$$

$$f(t) = \frac{E}{\pi} + \sum_{n=2,4,6,\ldots}^{\infty} \frac{2E}{\pi(n^2 - 1)} \cos \frac{n\pi t}{T} + \frac{E}{2} \sin \frac{\pi t}{T}$$

ANSWERS

1. $-1 + 2 \displaystyle\sum_1^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$

2. $\dfrac{4\pi^2}{3} + 4 \displaystyle\sum_1^{\infty} \frac{\cos nx}{n^2} - 4\pi \sum_1^{\infty} \frac{\sin nx}{n}$

3. $\dfrac{2\pi^2}{3} + 4 \displaystyle\sum_1^{\infty} \frac{(-1)^{n+1}}{n^2} \cos nx$

4. $\displaystyle\sum_1^{\infty} \frac{(-1)^{n+1}}{n^3} \sin nx$

5. $-1 - \frac{1}{2}\cos x + 2\sum_{2}^{\infty}\frac{\cos nx}{n^2-1} + \pi \sin x$

6. $\sum_{1}^{\infty} \frac{1+3(-1)^{n+1}}{n} \sin nx$

7. $\frac{\pi}{2} + \frac{4}{\pi} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^2} \cos nx$

8. $\frac{8}{\pi} \sum_{1}^{\infty} \frac{n}{n^2-1} \sin nx$

9. $\frac{4}{\pi} \sum_{1}^{\infty} \frac{\sin nx}{n}$

10. $\frac{4l^2}{3} + \frac{4l^2}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{l}$

11. $\frac{4}{\pi^2} \sum_{1}^{\infty} \frac{1}{n^2} \{1-(-1)^n\} \cos n\pi x$

12. $\frac{4}{3} + \frac{16}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2} + \frac{4}{\pi} \sum_{1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{2}$.

13. $\frac{1}{6} - \frac{1}{\pi^2} \sum_{1}^{\infty} \frac{1}{n^2} \cos 2n\pi x$

14. $l - \frac{4l}{\pi^2} \sum_{n=1,3,5,\cdots}^{\infty} \frac{1}{n^2} \cos \frac{n\pi x}{l}$

15. $\sum_{1}^{\infty} \left[\frac{2(-1)^{n+1}}{n\pi} - \frac{4}{n^3\pi^3} \{1-(-1)^n\} \right] \sin(n\pi x)$

16. $\frac{32}{\pi} \sum_{1}^{\infty} \frac{1}{n^2} \sin \frac{n\pi x}{2}$

17. $\frac{1}{4} + \frac{4}{\pi^2} \sum_{1}^{\infty} \frac{1}{n^2} \left\{ 1 - \cos \frac{n\pi}{2} \right\} \cos \frac{n\pi x}{2}$

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1.8 Half Range Fourier Series

In an interval of length $2l$ we have seen that in general a periodic function of x will have Fourier expansion containing cosine terms and sine terms. Many times it becomes necessary to have the expansion containing only cosine terms or only sine terms.

To achieve this, the function must be defined in the interval of the form $(0, l)$ which is to be regarded as half the interval. We then extend the definition to the other half in such a manner that the function becomes even or odd. This will result in cosine series only or sine series only.

Case-(i) : ***For cosine series***

$$f(x) = \begin{cases} \phi(x) & \text{in } (0, l) \text{ as given} \\ \phi(-x) & \text{in } (-l, 0) \ [\textit{assumed}] \text{ to make } f(x) \text{ even} \end{cases}$$

Case-(ii) : ***For sine series***

$$f(x) = \begin{cases} \phi(x) & \text{in } (0, l) \text{ as given} \\ -\phi(-x) & \text{in } (-l, 0) \ [\textit{assumed}] \text{ to make } f(x) \text{ odd} \end{cases}$$

We have already seen that in the case-(i) $b_n = 0$ and in the case - (ii) $a_0 = 0, a_n = 0$

$\therefore$ we have in the case - (i), $f(x) = \dfrac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \dfrac{n\pi x}{l}$ . . . (1)

where $a_0 = \dfrac{2}{l} \int_0^l f(x)\, dx$, $a_n = \dfrac{2}{l} \int_0^l f(x) \cos \dfrac{n\pi x}{l}\, dx$

Also in the case (ii), $f(x) = \sum_{n=1}^{\infty} b_n \sin \dfrac{n\pi x}{l}$ . . . (2)

where $b_n = \dfrac{2}{l} \int_0^l f(x) \sin \dfrac{n\pi x}{l}\, dx$

The series (1) which contains only cosine terms is called the ***cosine half range Fourier series*** for $f(x)$ in $(0, l)$ and the series (2) which contains only sine terms is called the ***sine half range Fourier series*** for $f(x)$ in $(0, l)$

Similar consideration hold good for $(0, \pi)$ as it is a particular case when $l = \pi$.

The following table summarizes the theory discussed and will be useful for working problems.

$f(x)$ in	Required Series	Series	Fourier Coefficients
$(0, l)$	Cosine series	$\frac{a_0}{2} + \sum_1^{\infty} a_n \cos \frac{n\pi x}{l}$	$a_0 = \frac{2}{l} \int_0^l f(x)\, dx$ $a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx$
$(0, l)$	Sineseries	$\sum_1^{\infty} b_n \sin \frac{n\pi x}{l}$	$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l}\, dx$
$(0, \pi)$	Cosine series	$\frac{a_0}{2} + \sum_1^{\infty} a_n \cos nx$	$a_0 = \frac{2}{\pi} \int_0^{\pi} f(x)\, dx$ $a_n = \frac{2}{\pi} \int_0^{\pi} f(x) \cos nx\, dx$
$(0, \pi)$	Sine series	$\sum_1^{\infty} b_n \sin nx$	$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx\, dx$

Illustrations : **(i)** Suppose we want to find the cosine half range Fourier series for $f(x) = l - x$ in $(0, l)$ Treating this as $\phi(x)$ in $(0, l)$ we take

$\psi(x) = \phi(-x)$ in $(-l, 0)$ so that we have

$$f(x) = \begin{cases} \phi(x) = l - x \text{ in } (0, l) \text{ as given} \\ \psi(x) = l + x \text{ in } (-l, 0) \text{ assumed} \end{cases}$$

It may to observed that $f(x)$ is even in $(-l, l)$. Consequently $b_n = 0$ and the Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n \cos \frac{n\pi x}{l} \qquad \ldots(1)$$

where $a_0 = \frac{2}{l} \int_0^l f(x)\, dx = \frac{2}{l} \int_0^l (l - x)\, dx$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \frac{n\pi x}{l}\, dx = \frac{2}{l} \int_0^l (l - x) \cos \frac{n\pi x}{l}\, dx$$

The series represented by (1) on substitution of the values of a_0 and a_n is the required cosine half range Fourier series for $f(x)$ in $(0, l)$.

(ii) Suppose we want to find the sine half range Fourier series for $f(x) = x^2$ in $0 < x < \pi$. Treating this as $\phi(x)$ in $(0, \pi)$ we take
$\psi(x) = -\phi(-x)$ in $(-\pi, 0)$ so that we have

$$f(x) = \begin{cases} \phi(x) = x^2 \text{ in } (0, \pi) \text{ as given} \\ \psi(x) = -(-x)^2 = -x^2 \text{ in } (-\pi, 0) \text{ assumed} \end{cases}$$

It may be observed that $f(x)$ is odd in $(-\pi, \pi)$. Consequently $a_0 = 0$, $a_n = 0$ and the Fourier series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \qquad \ldots (2)$$

where $b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x) \sin nx \, dx = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x^2 \sin nx \, dx$

The series represented by (2) on substitution of the value of b_n is the required sine half range series.

Note : *In problems of half range Fourier series we can directly assume the appropriate series along with the expressions for the Fourier coefficients as summarized in the table before.*

WORKED PROBLEMS

31. *Obtain the sine half range Fourier series of* $f(x) = x^2$ *in* $0 < x < \pi$

>> The sine half range Fourier series of the function $f(x)$ in $(0, \pi)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx \quad \text{where } b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

We have, $\quad b_n = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} x^2 \sin nx \, dx.$ Applying Bernoulli's rule,

$$b_n = \frac{2}{\pi}\left[x^2 \cdot -\frac{\cos nx}{n} - 2x \cdot -\frac{\sin nx}{n^2} + 2 \cdot \frac{\cos nx}{n^3} \right]_0^{\pi}$$

$$= \frac{2}{\pi}\left\{ \frac{-1}{n}\left[x^2 \cos nx \right]_0^{\pi} + 0 + \frac{2}{n^3}\left[\cos nx \right]_0^{\pi} \right\}$$

$$= \frac{2}{\pi}\left\{ \frac{-1}{n}(\pi^2 \cos n\pi - 0) + \frac{2}{n^3}(\cos n\pi - 1) \right\}$$

$$b_n = \frac{2}{\pi}\left\{ \frac{(-1)^{n+1}\pi^2}{n} - \frac{2}{n^3}[1 - (-1)^n] \right\}$$

Thus the required sine half range Fourier series is given by

$$f(x) = \sum_{1}^{\infty} \frac{2}{\pi} \left\{ \frac{(-1)^{n+1} \pi^2}{n} - \frac{2}{n^3} [1-(-1)^n] \right\} \sin nx$$

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32. *Expand* $f(x) = 2x - 1$ *as a cosine half range Fourier series in* $0 < x < 1$

>> Comparing the given interval $(0, 1)$ with $(0, l)$ we have $l = 1$. The corresponding cosine half range Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

where $a_0 = \frac{2}{1} \int_0^1 f(x)\, dx, \quad a_n = \frac{2}{1} \int_0^1 f(x) \cos n\pi x\, dx$

$$a_0 = 2 \int_0^1 (2x-1)\, dx = 2\Big[x^2 - x \Big]_0^1 = 0$$

$$a_n = 2 \int_0^1 (2x-1) \cos n\pi x\, dx.$$

$$= 2 \left[(2x-1) \frac{\sin n\pi x}{n\pi} - (2) \cdot \frac{-\cos n\pi x}{n^2 \pi^2} \right]_0^1$$

$$= \frac{4}{n^2 \pi^2} \Big[\cos n\pi x \Big]_0^1 = \frac{4}{n^2 \pi^2} (\cos n\pi - 1)$$

$$a_n = \frac{-4}{n^2 \pi^2} \{1-(-1)^n\}$$

Thus the required cosine half range Fourier series is given by

$$f(x) = \sum_{1}^{\infty} \frac{-4}{n^2 \pi^2} \{1-(-1)^n\} \cos n\pi x$$

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33. *Show that the sine half range series for the function* $f(x) = lx - x^2$ *in* $0 < x < l$ *is*

$$\frac{8l^2}{\pi^3} \sum_{0}^{\infty} \frac{1}{(2n+1)^3} \sin \left(\frac{2n+1}{l} \right) \pi x$$

>> The sine half range Fourier series of $f(x)$ in $(0, l)$ is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where}$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} \, dx$$

$$= \frac{2}{l} \int_0^l (lx - x^2) \sin \frac{n\pi x}{l} \, dx.$$ Applying Bernoulli's rule,

$$= \frac{2}{l} \left[(lx - x^2) \cdot \frac{-\cos \frac{n\pi x}{l}}{(n\pi/l)} - (l - 2x) \cdot \frac{-\sin \frac{n\pi x}{l}}{(n\pi/l)^2} + (-2) \frac{\cos \frac{n\pi x}{l}}{(n\pi/l)^3} \right]_0^l$$

$$= \frac{-4}{l} \cdot \frac{l^3}{n^3 \pi^3} \left[\cos \frac{n\pi x}{l} \right]_0^l$$

($lx - x^2$ is zero at $x = 0$, l and $\sin n\pi = 0 = \sin 0$)

$$b_n = \frac{-4l^2}{n^3 \pi^3} (\cos n\pi - 1) = \frac{4l^2}{n^3 \pi^3} \{1 - (-1)^n\}$$

The sine half range Fourier series is given by

$$f(x) = \frac{4l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \{1 - (-1)^n\} \sin \frac{n\pi x}{l}$$

But $$1 - (-1)^n = \begin{cases} 1 - (+1) = 0 \text{ when } n \text{ is even} \\ 1 - (-1) = 2 \text{ when } n \text{ is odd} \end{cases}$$

$\therefore$ $$f(x) = \frac{4l^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{2}{n^3} \sin \frac{n\pi x}{l}$$

ie., $$f(x) = \frac{8l^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{l}$$

But 1, 3, 5, . . . are odd numbers represented in general as $(2n+1)$ where $n = 0, 1, 2, 3, \ldots$. Thus we have,

$$f(x) = \frac{8l^2}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^3} \sin\left(\frac{2n+1}{l}\right) \pi x$$